1. Find the singular value decomposition of the matrices.

a.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

b.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

solution.

a. We can obtain singular value and right-singular vector matrix $\boldsymbol{V}$ from eigenvalue decomposition of $\boldsymbol{A}^\top \boldsymbol{A}$.

$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^\top$$

We can obtain singular values and right-singular vector matrix.

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}$$

We can find the left-singular vectors with corresponding singular value and right-singular vector.

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} \boldsymbol{A} \boldsymbol{v}_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} \boldsymbol{A} \boldsymbol{v}_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence left-singular vector matrix $\boldsymbol{U}$ is

$$\boldsymbol{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and singular value decomposition of matrix $\boldsymbol{A}$ is as follows.

$$\boldsymbol{A} = \boldsymbol{U} \Sigma \boldsymbol{V}^\top = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \end{bmatrix}^\top$$

b. As problem a., first calculate an eigenvalue decomposition of the $\boldsymbol{A}^\top \boldsymbol{A}$.

$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^\top$$

Therefore, singular values and right-singular vector matrix are

$$\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Similarly, left-singular vectors are as follows.

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1}\boldsymbol{A}\boldsymbol{v}_1 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2}\boldsymbol{A}\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Finally, singular value decomposition of $\boldsymbol{A}$ is

$$\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^\top = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^\top$$

5 points: Correct solution and answer
4 points: Correct answer without solution
4 points: Correct answer / Calculate $\boldsymbol{U}$ and $\boldsymbol{V}$ independently
4 points: Correct solution but minor mistake e.g. calculation
3 points: Wrong answer by calculating $\boldsymbol{U}$ and $\boldsymbol{V}$ independently

2. Show that for any $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ the matrices $\boldsymbol{A}^\top\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^\top$ possess the same nonzero eigenvalues.

solution. Let λ be a nonzero eigenvalue of $\boldsymbol{A}^\top\boldsymbol{A}$ and $\boldsymbol{v}$ become corresponding eigenvector. From $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, $\boldsymbol{A}\boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{v}) = \lambda(\boldsymbol{A}\boldsymbol{v})$. Here, $\boldsymbol{A}\boldsymbol{v} \neq 0$ since $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$ implies $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} = 0$ which is contradiction. Hence λ is also an eigenvalue of $\boldsymbol{A}\boldsymbol{A}^\top$ with eigenvector $\boldsymbol{A}\boldsymbol{v}$. Opposite direction is also satisfied in the same way.

-1 point if proved for single direction
-1 point for unclear statements
-0 points if did not prove $\boldsymbol{A}\boldsymbol{v} \neq 0$

3. Consider the quadratic program,

$$\min_{x\in\mathbb{R}^2} \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Derive the dual quadratic program using Lagrange duality.

solution. Lets denote the quadratic program as:

$$\min_{x \in \mathbb{R}^d} \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x}$$

subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. Then Lagrangian is

$$\mathfrak{L}(\boldsymbol{x}, \lambda) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x} + \lambda^\top(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$$

To find the Lagrange dual problem, we should differentiate the Lagrangian with respect to x, set the differential to zero, and solve for the optimal value.

$$\boldsymbol{Q}\boldsymbol{x} + (\boldsymbol{c} + \boldsymbol{A}^\top \lambda) = 0$$

$$\boldsymbol{x} = -\boldsymbol{Q}^{-1}(\boldsymbol{c} + \boldsymbol{A}^\top \lambda)$$

Then dual Lagrangian is

$$\mathfrak{D}(\lambda) = -\frac{1}{2}(\boldsymbol{c} + \boldsymbol{A}^\top \lambda)^\top \boldsymbol{Q}^{-1}(\boldsymbol{c} + \boldsymbol{A}^\top \lambda) - \lambda^\top \boldsymbol{b}$$

and dual quadratic program is

$$\max_{\lambda \in \mathbb{R}^4} -\frac{1}{2}(\boldsymbol{c} + \boldsymbol{A}^\top \lambda)^\top \boldsymbol{Q}^{-1}(\boldsymbol{c} + \boldsymbol{A}^\top \lambda) - \lambda^\top \boldsymbol{b}$$

subject to $\lambda \geq 0$.

-1 point if minor mistake

4. Consider the negative entropy of $x \in \mathbb{R}^D$,

$$f(\mathbf{x}) = \sum_{d=1}^{D} x_d \log x_d$$

Derive the convex conjugate function $f^*(\mathbf{y})$, by assuming the standard dot product.

solution. From the definition of convex conjugate,

$$f^*(\mathbf{y}) = \sup_{\mathbf{x}}\{\langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x})\}$$

$$= \sup_{\mathbf{x}} \sum_{d=1}^{D} (x_d y_d - x_d \log x_d)$$

We can find the maximum by taking the derivative and with respect to x_d setting it to zero.

$$\frac{\partial}{\partial x_d}(x_d y_d - x_d \log x_d) = y_d - \log x_d - 1 = 0$$

$$x_d = e^{y_d - 1}$$

Here x_d is a maximizer since derivative is monotone decreasing. Hence convex conjugate function is

$$f^*(\mathbf{y}) = \sum_{d=1}^{D} (y_d e^{y_d - 1} - (y_d - 1)e^{y_d - 1})$$

$$= \sum_{d=1}^{D} e^{y_d - 1}.$$

-1 point if minor mistake

5. Consider the function

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \boldsymbol{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c,$$

where $\boldsymbol{A}$ is strictly positive definite, which means that it is invertible. Derive the convex conjugate of $f(\mathbf{x})$.

solution. From the definition of convex conjugate,

$$\begin{aligned} f^*(\mathbf{y}) &= \sup_{\mathbf{x}}\{\langle \mathbf{y}, \mathbf{x}\rangle - f(\mathbf{x})\} \\ &= \sup_{x}\{-\frac{1}{2}\mathbf{x}^\top \boldsymbol{A}\mathbf{x} + (\mathbf{y} - \mathbf{b})^\top \mathbf{x} - c\} \end{aligned}$$

We can find the maximum by taking the derivative and with respect to x_d setting it to zero.

$$\nabla_{\mathbf{x}}\{-\frac{1}{2}\mathbf{x}^\top \boldsymbol{A}\mathbf{x} + (\mathbf{y} - \mathbf{b})^\top \mathbf{x} - c\} = -\mathbf{A}\mathbf{x} + \mathbf{y} - \mathbf{b} = 0$$

$$\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$$

Hence convex conjugate function is

$$f^*(\mathbf{y}) = \frac{1}{2}(\mathbf{y} - \mathbf{b})^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) - c.$$

-1 point if minor mistake