Homework 1: Mathematics for AI
Due date: September 28th, 11:59 pm.
Note: Submit your solution file to TA, Minkyu Kim (e-mail: kimmk135@kaist.ac.kr), by email. For questions, contact to Minkyu as well.

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1. Find the singular value decomposition of the matrices.

a.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

b.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

2. Show that for any $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ the matrices $\boldsymbol{A}^\top \boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^\top$ possess the same nonzero eigenvalues.

3. Consider the quadratic program,

$$\min_{x \in \mathbb{R}^2} \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Derive the dual quadratic program using Lagrange duality.

4. Consider the negative entropy of $x \in \mathbb{R}^D$,

$$f(\mathbf{x}) = \sum_{d=1}^{D} x_d \log x_d$$

Derive the convex conjugate function $f^*(\mathbf{y})$, by assuming the standard dot product.

5. Consider the function

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \boldsymbol{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c,$$

where $\boldsymbol{A}$ is strictly positive definite, which means that it is invertible. Derive the convex conjugate of $f(\mathbf{x})$.