Message Passing Algorithms: Communication, Inference and Optimization

Jinwoo Shin

KAIST EE

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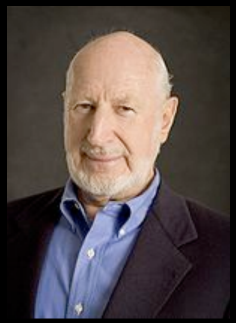

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(1970s)

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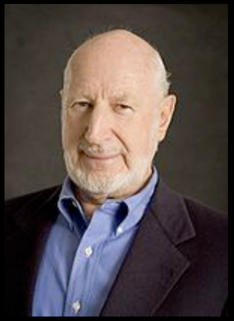

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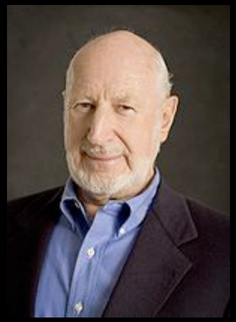

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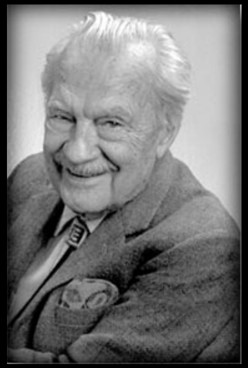

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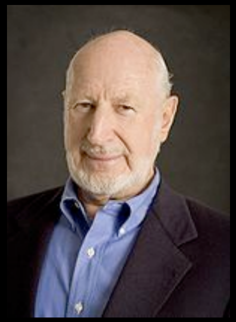

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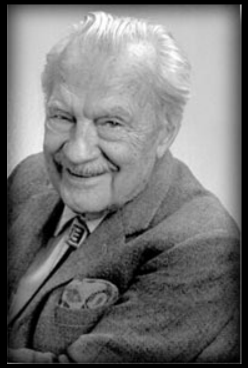

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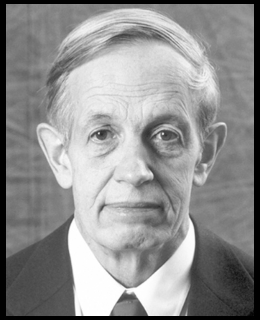

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Part I of This Talk

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Part II of This Talk

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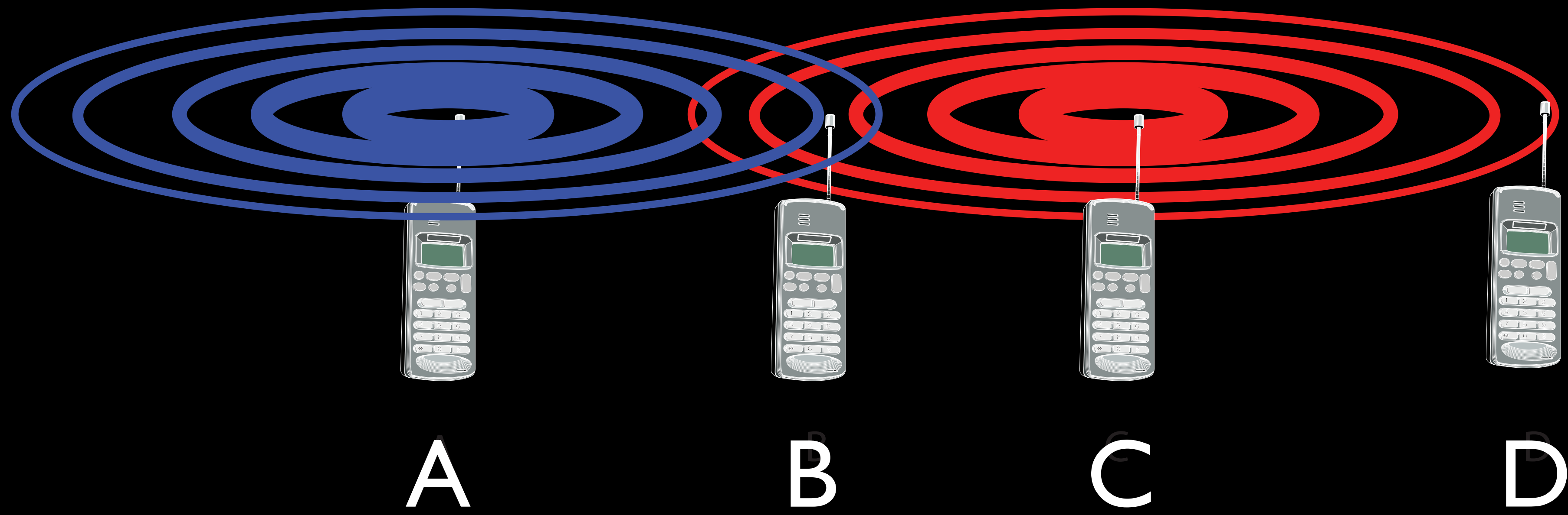
- Part I -
Message Passing in Communication Networks

- Focus on medium access for wireless networks
  Joint work with Devavrat Shah (MIT)
- Later describe how the result is extended
  Joint work with Yung Yi (KAIST), Seyoung Yun,(MSR), Tonghoon Suk (Gatech)

Motivation : Wireless Network



- A and C cannot send packets to B simultaneously
  - 'A->B' and 'C->B' interfere with each other & packets collide

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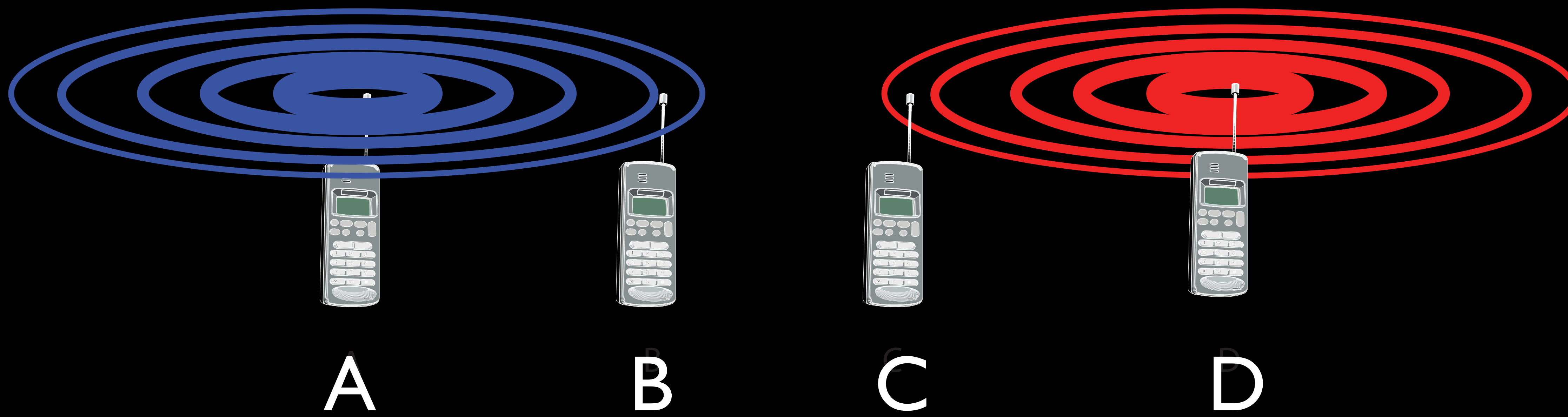

- However, ‘A->B’ and ‘D->C’ do not interfere with each other

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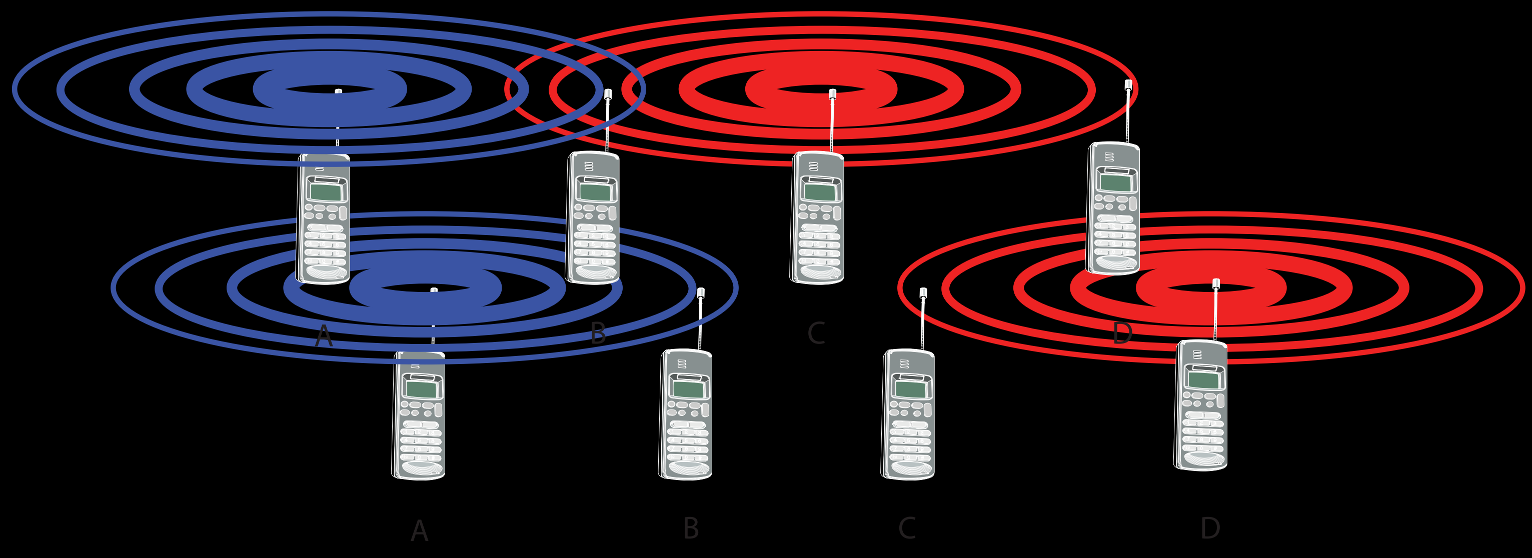

- Question : How to avoid interference?
  - While transmitting as many packets as possible

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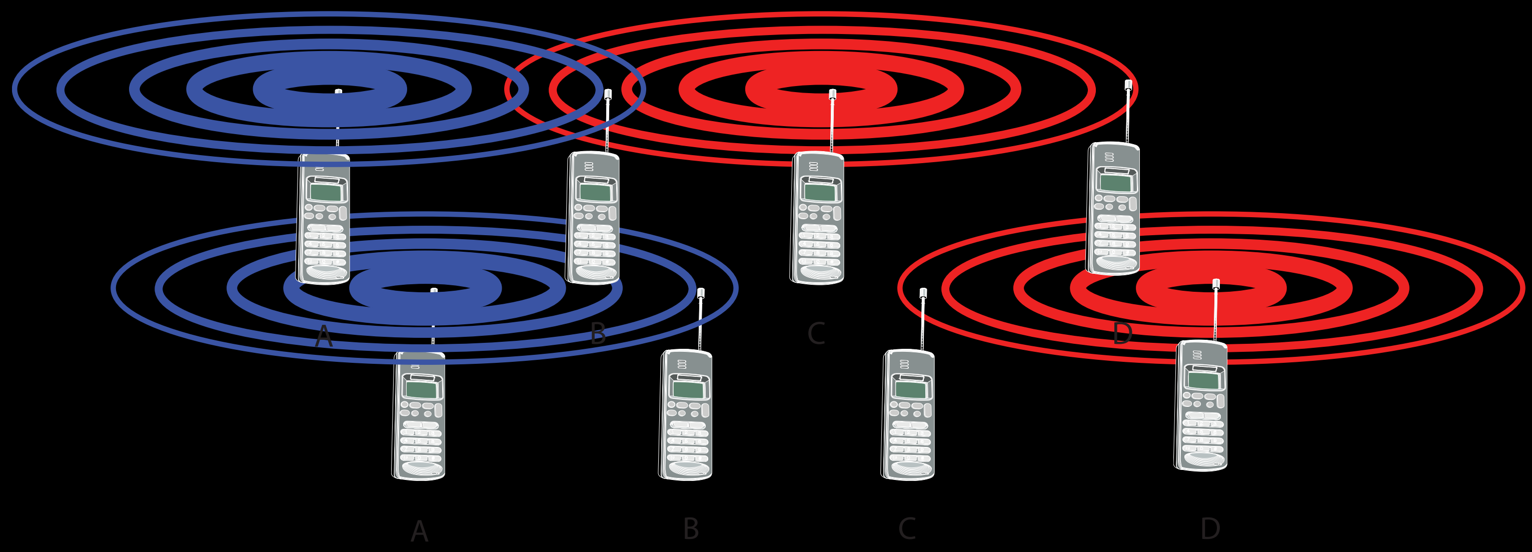

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- Need a contention resolution protocol
  - Also called medium access algorithm
  - e.g. CSMA/CA, ALOHA, TDMA, CDMA, etc.

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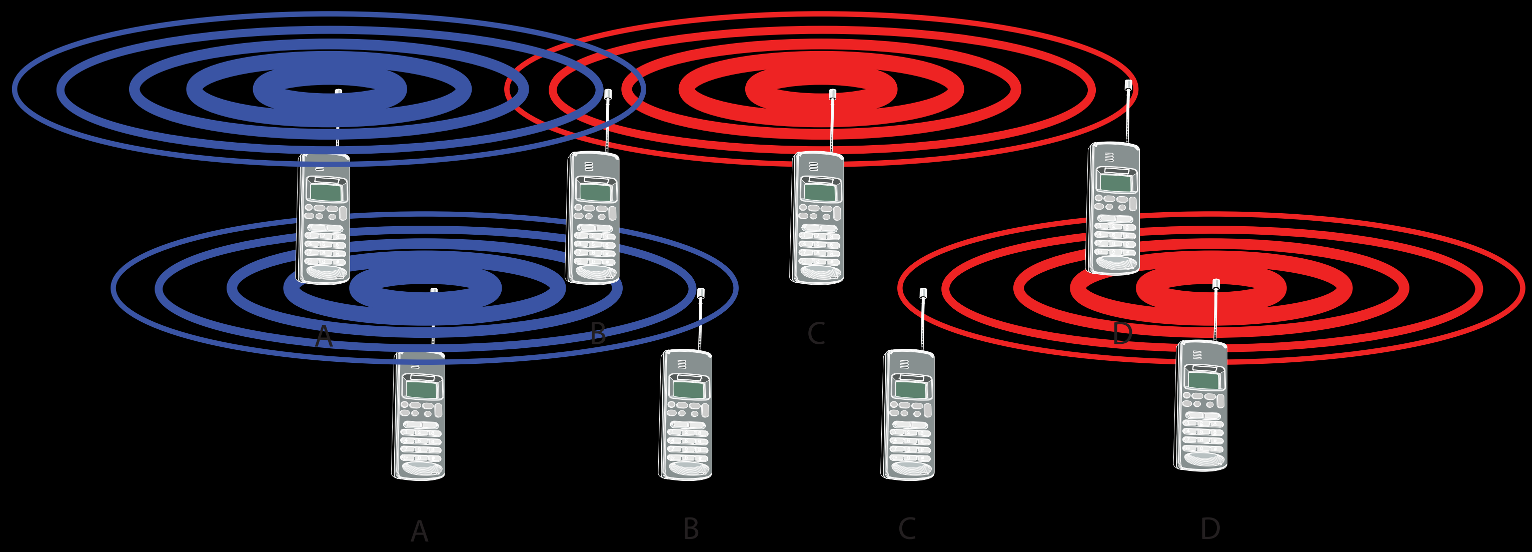

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However, no protocol is known to be optimal in a certain sense

Our Goal

- Design a `provably' optimal medium access algorithm

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Next : Describe a mathematical model

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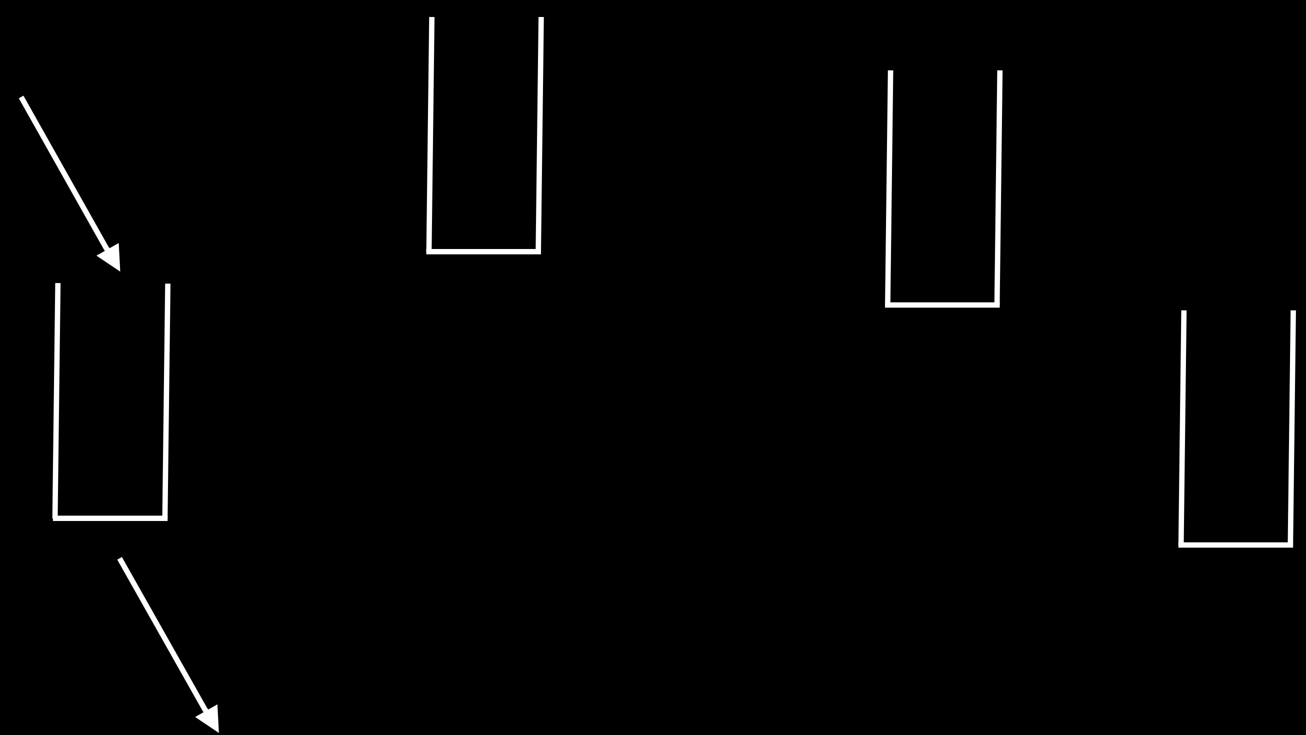
Next : Definition of `optimality’

  - For simplicity, I will considers a simple model (i.e. discrete-time, single-hop, single-channel)
  - However, the same story goes through for other models (i.e. continuous-time,multi-hop, multi-channel, collisions, time-varying)

Model : Wireless Network



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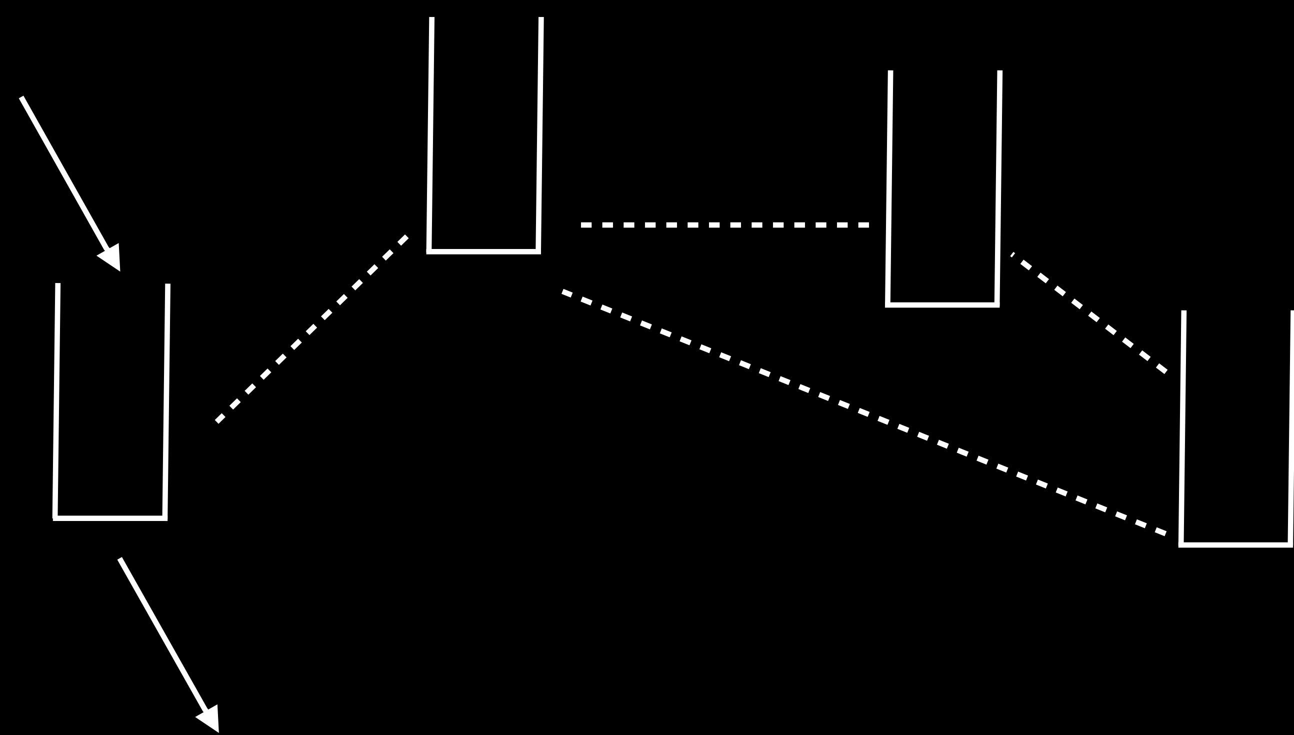

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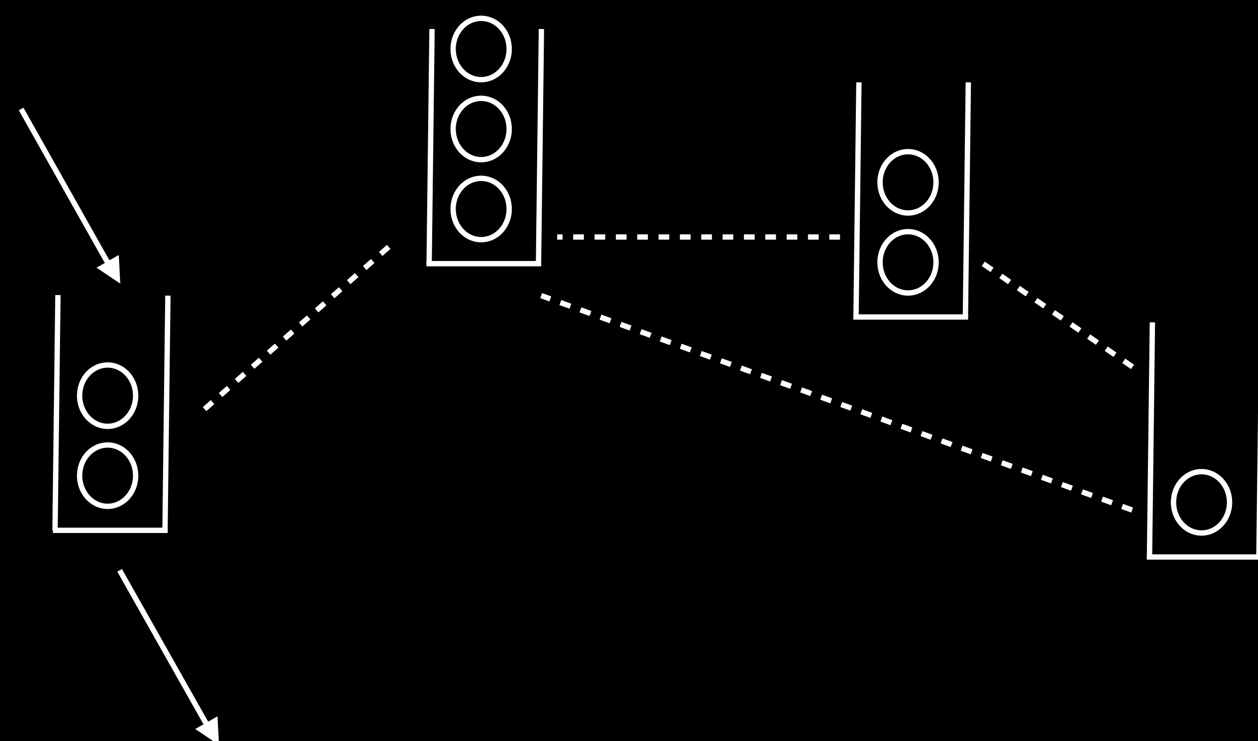
  - Interference graph G

"Queues form vertices & two queues share an edge if they cannot transmit simultaneously"

Model : Wireless Network



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  - Packets arrive at queue i with rate $\lambda(i)$ e.g. Bernoulli stochastic process
  - Time is discrete & at most one packet can depart from each queue at each time instance

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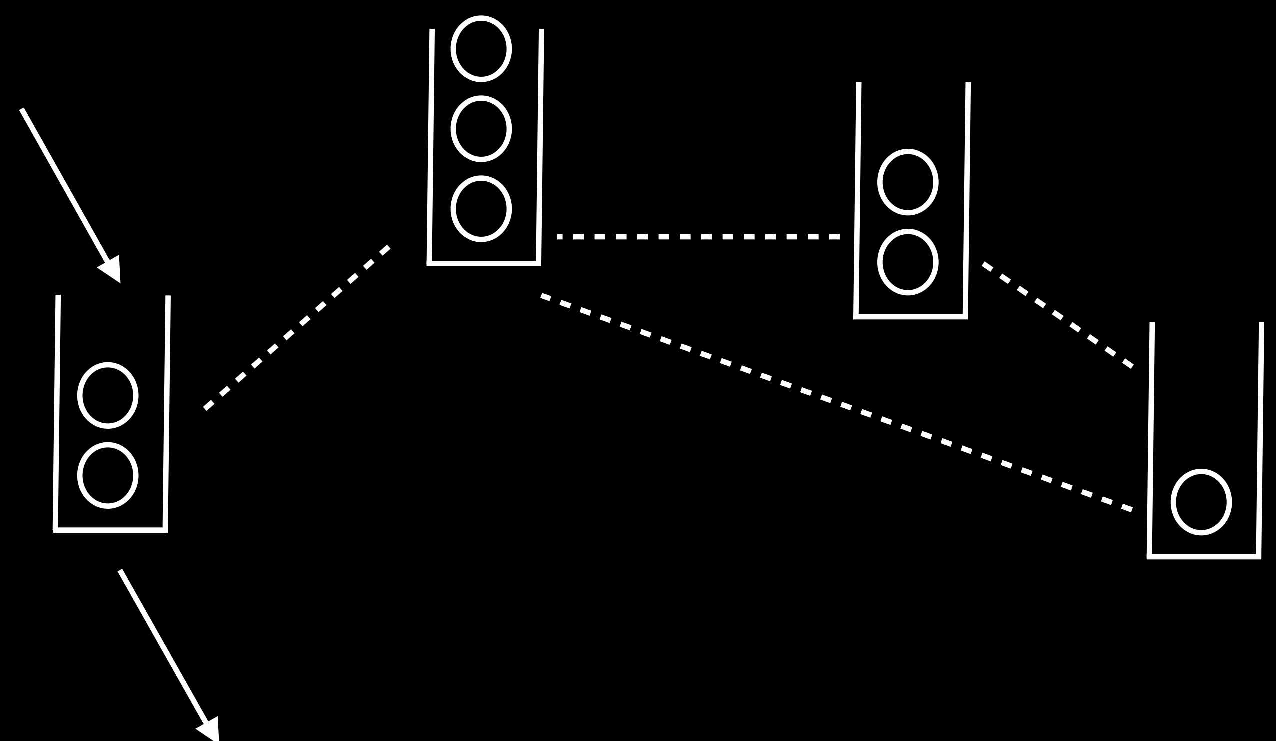

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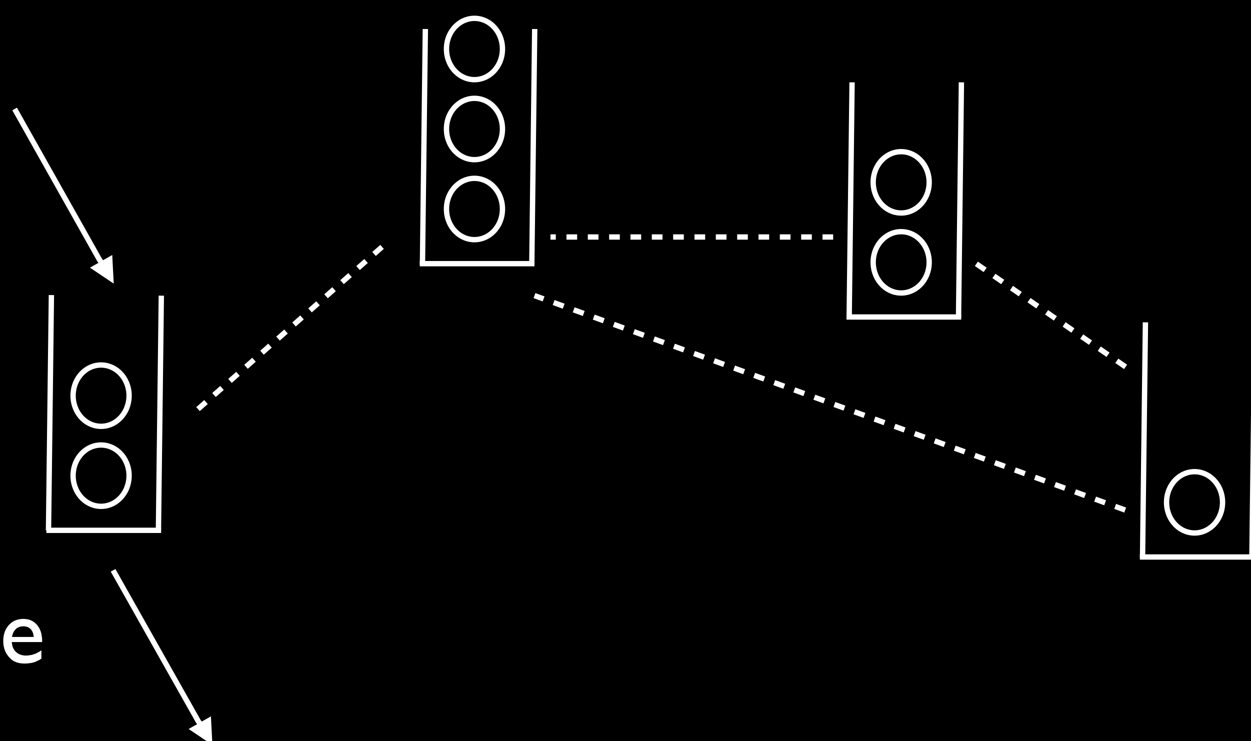

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- Next : Example of simple medium access algorithm

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- Each individual queue attempts to transmit at time t
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Open Question for General Interference Graph

- Much Research starting from1970s in various setups

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  - Till 1990s : complete interference graph (e.g. Ethernet)
  - From 1990s : general interference graph (e.g. wireless networks)

- No simple distributed throughput-optimal protocol is known

  for **general** interference graph

  - Next : Recent progress for this open question

Medium Access using Carrier Sensing

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  - Knowledge whether neighbors attempted to transmit (at the previous time instance)
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How to design this `access probability' $p_i(t)$ for each queue i ?

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- [Jiang, Shah, S. and Walrand 2008]* provide provable T and ε
- Further improvements have been made, for example
  - [Liu, Yi, Proutiere, Chiang and Poor 2009] proved that even T=1 works.
  - [Chaporkar and Proutiere 2013] developed an algorithm even in SNR model
  - [Lee, Lee, Yi, Chong, Nardelli, Knightly and Chiang 2013] implemented them in 802.11

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How to Choose Access Probability in CSMA : Queue-based

$$\frac{1}{f(Q_i(t))}$$

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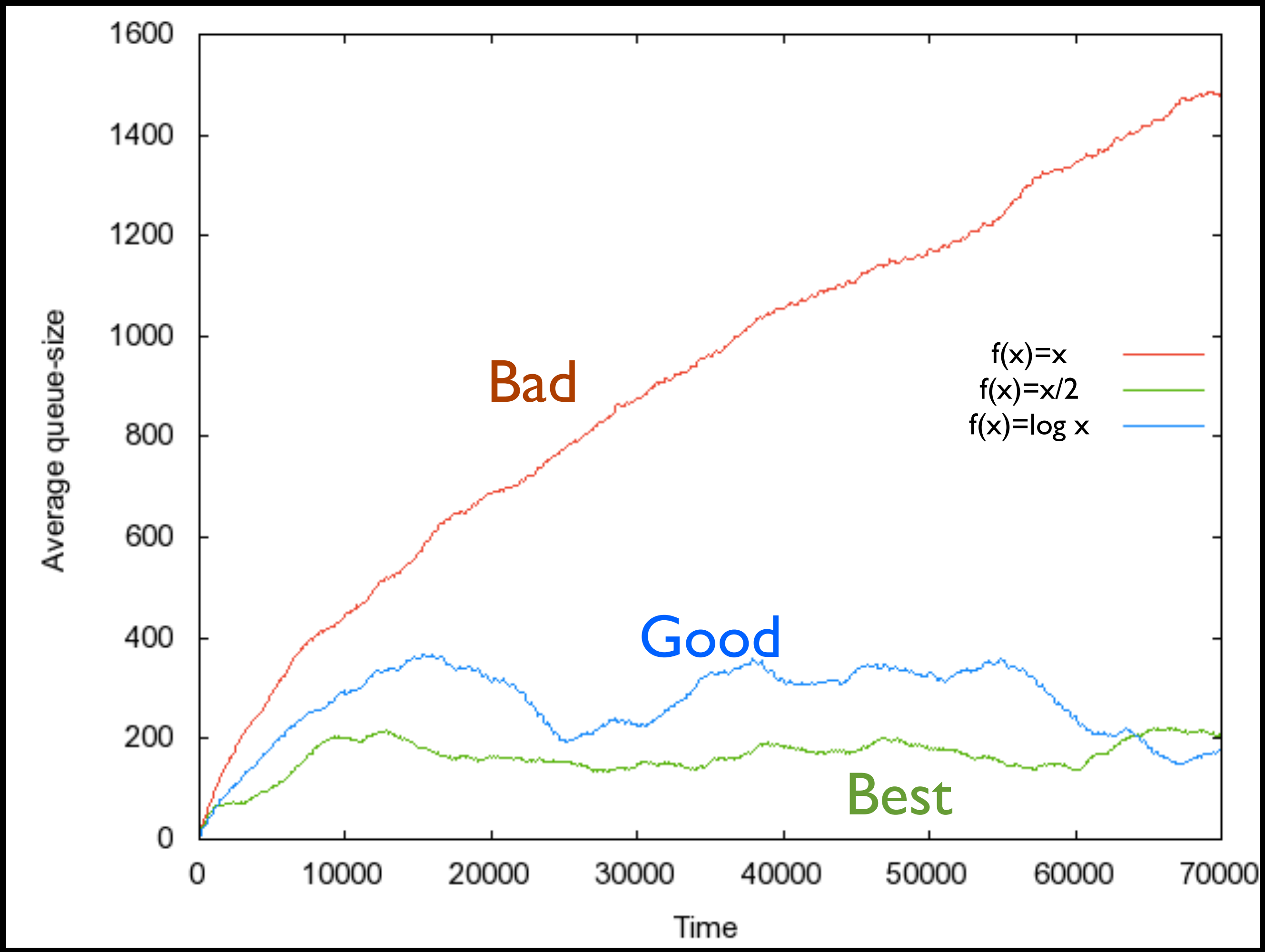
  - Simulations on 10 by 10 grid graph



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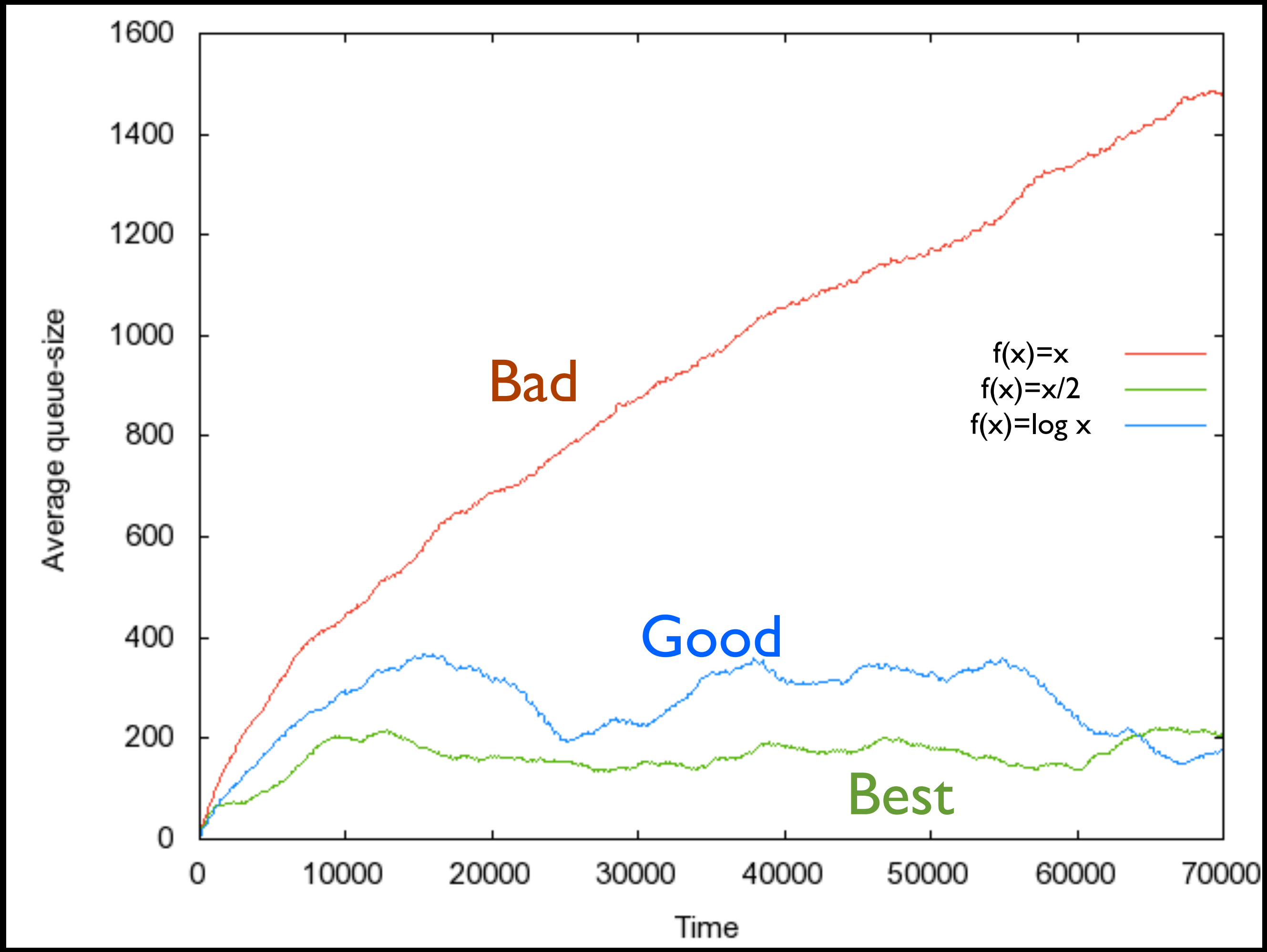
Which function f is best ?



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- Two recent approaches

	Rate-based CSMA	Queue-based CSMA
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  Queue-based CSMA with $f(x) \approx \log x$ is

  throughput-optimal for **general** interference graph

  - If any (even centralized) other algorithm can stabilize the network, this algorithm can also do it

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  - Next : Why we choose $f(x) \approx \log x$ for the positive recurrence ?

* Medium Access using Queues [Shah, Shin and Prasad] IEEE Symposium on Foundations of Computer Science 2011
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Network Adiabatic Theorem

- Theorem [Shah and S. 2009, 2010, 2011]*

  Queue-based CSMA with $f(x) \approx \log x$ is

  throughput-optimal for **general** interference graph

  - If any (even centralized) other algorithm can stabilize the network, this algorithm can also do it
  - We show positive recurrence (i.e. stability) of underlying network Markov chain

- Proof requires
  - Queueing theory (e.g. Lyapunov-Foster theorem)
  - Information theory (e.g. Gibbs maximal principle)
  - Spectral theory for matrices (e.g. Cheeger's inequality)
  - Combinatorics (e.g. Markov chain tree theorems)
  - Martingales (e.g. Doob's optional sampling theorem)

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why not $f(x)=x$, $x^{1/2}$ or loglog x ?

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Proof Strategy for Positive Recurrence

- Network MC (Markov Chain) $X(t)=\{Q(t), A(t)\}$
  - $Q(t)=[Q_i(t)]$: Vector of queue-sizes at time t
  - $A(t)=[A_i(t)]\in\{0,1\}^n$: Vector of attempting status at time t

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Which function is best for throughput and delay?

- Any sub-poly function works for throughput-optimality
  - Growing slower than x^{ε} e.g. $\log \mathbf{x}, \log\log \mathbf{x}, \mathbf{e}^{\sqrt{\log \mathbf{x}}}$ ...

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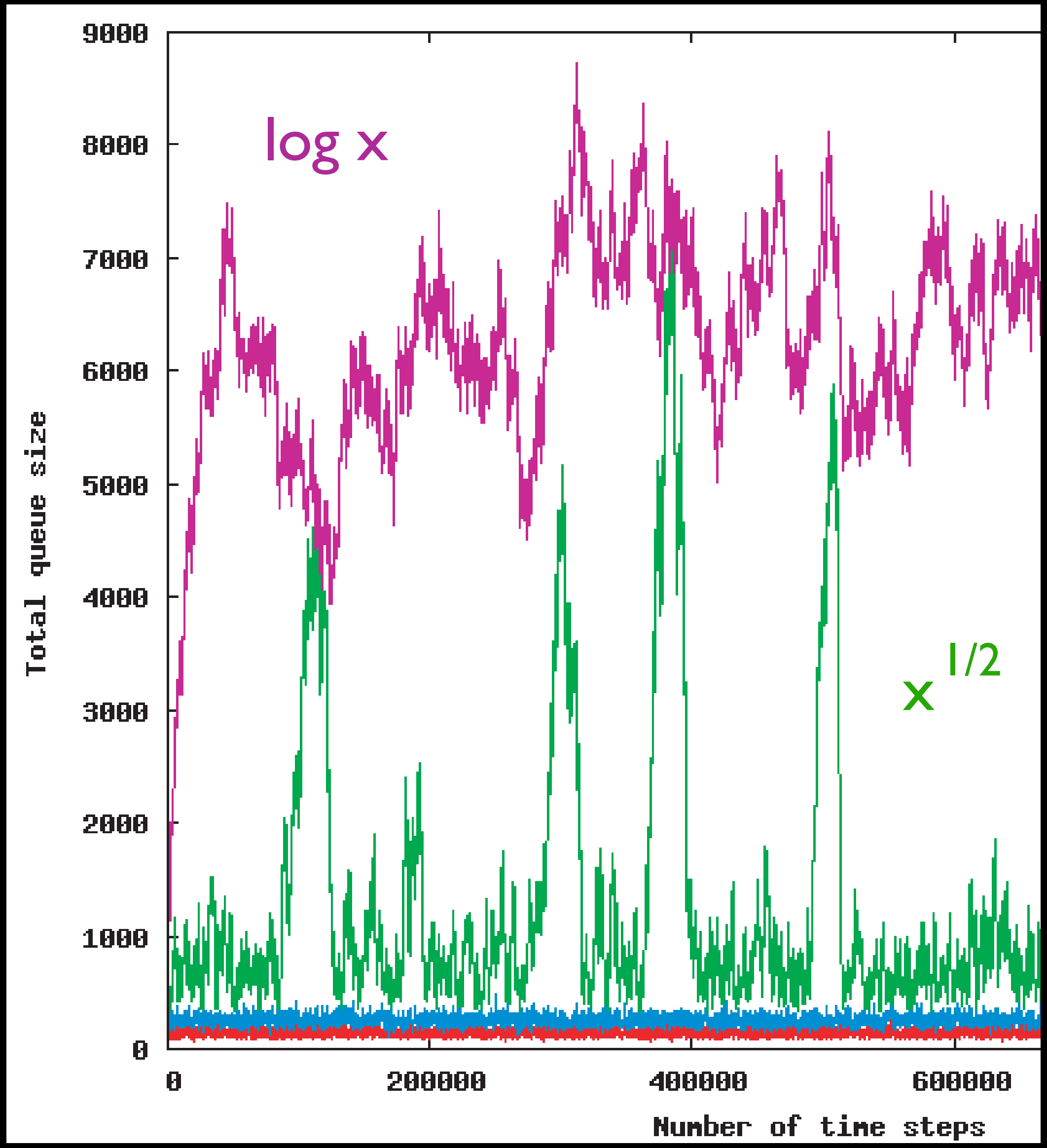

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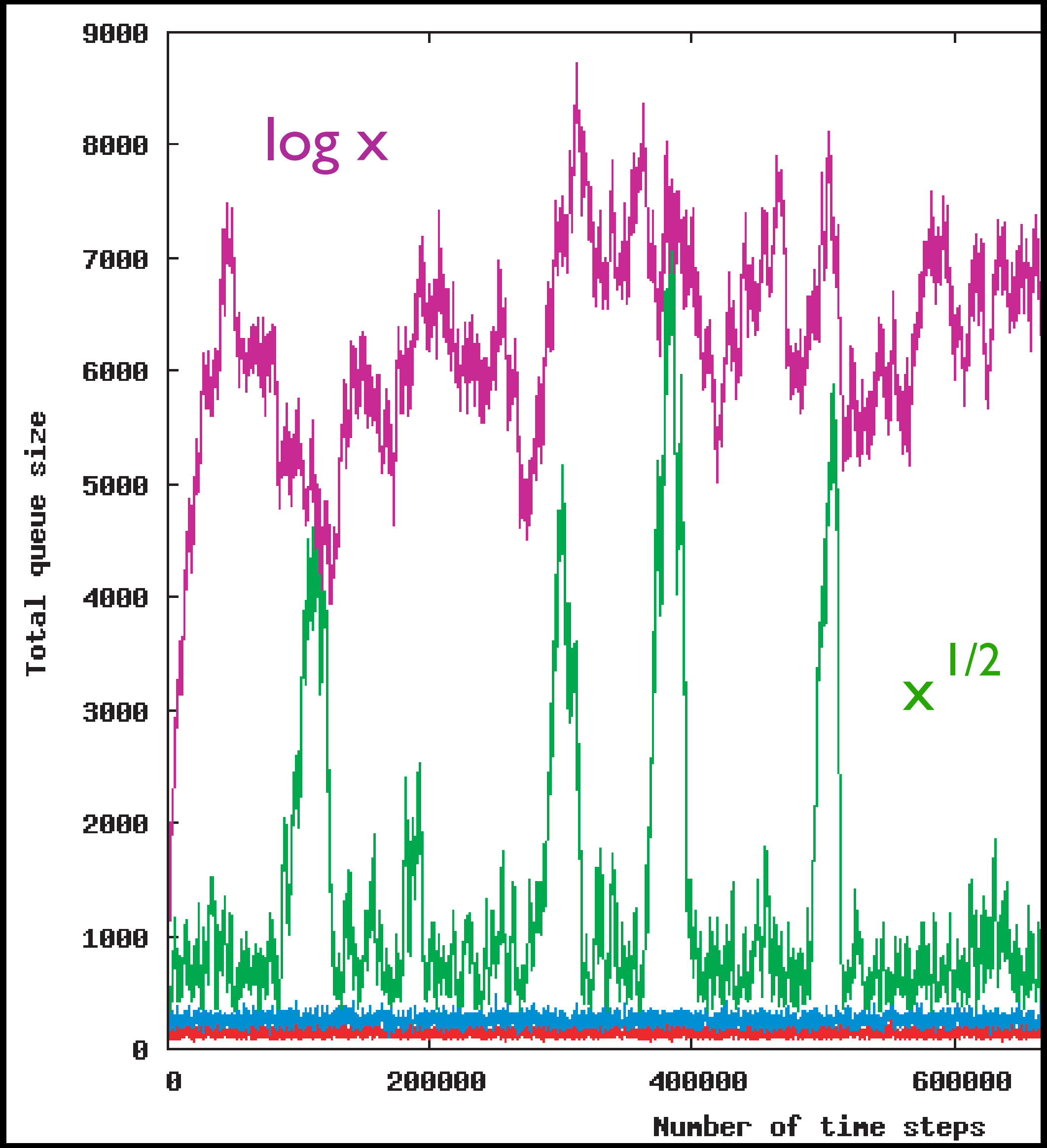

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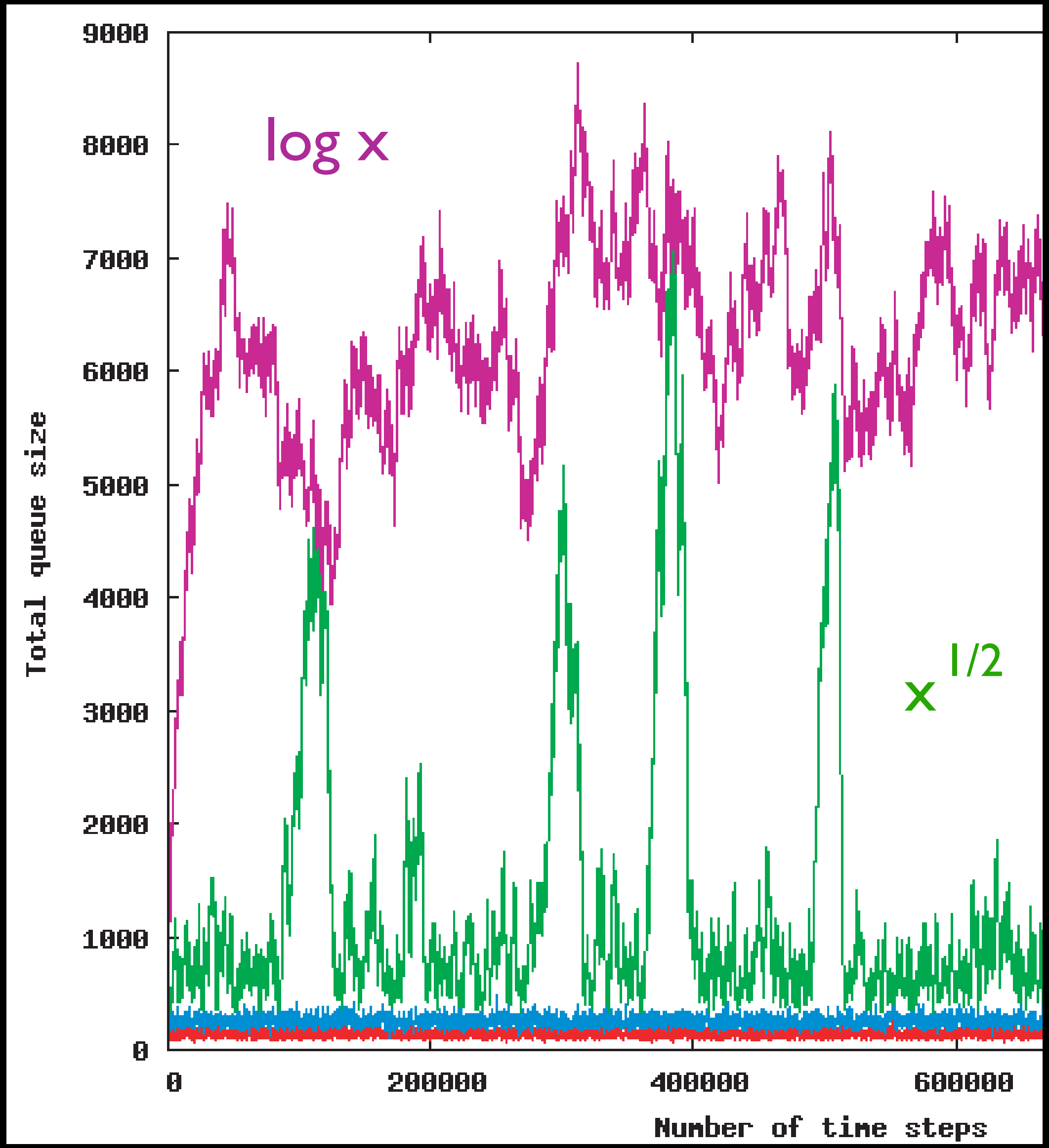

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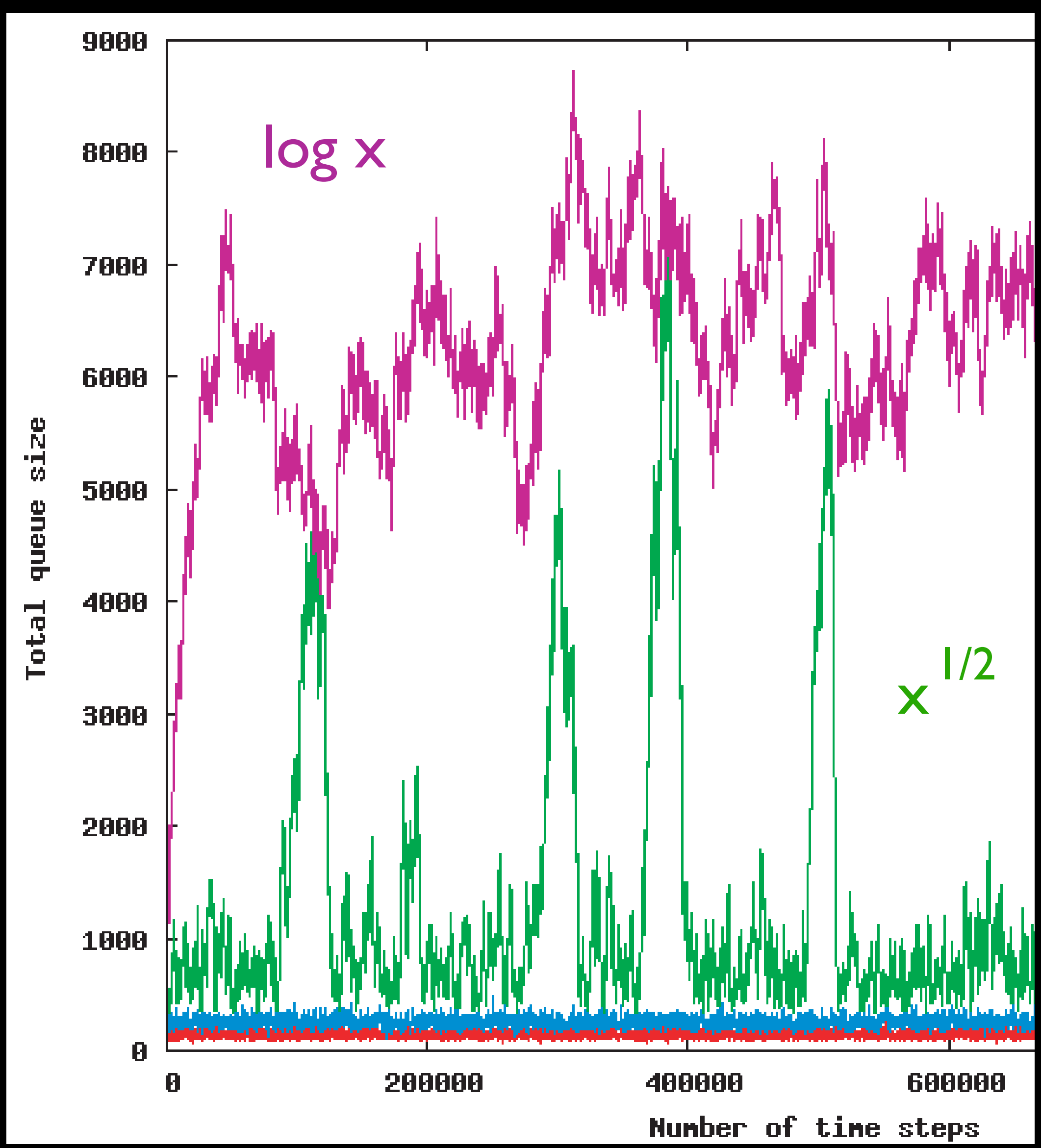

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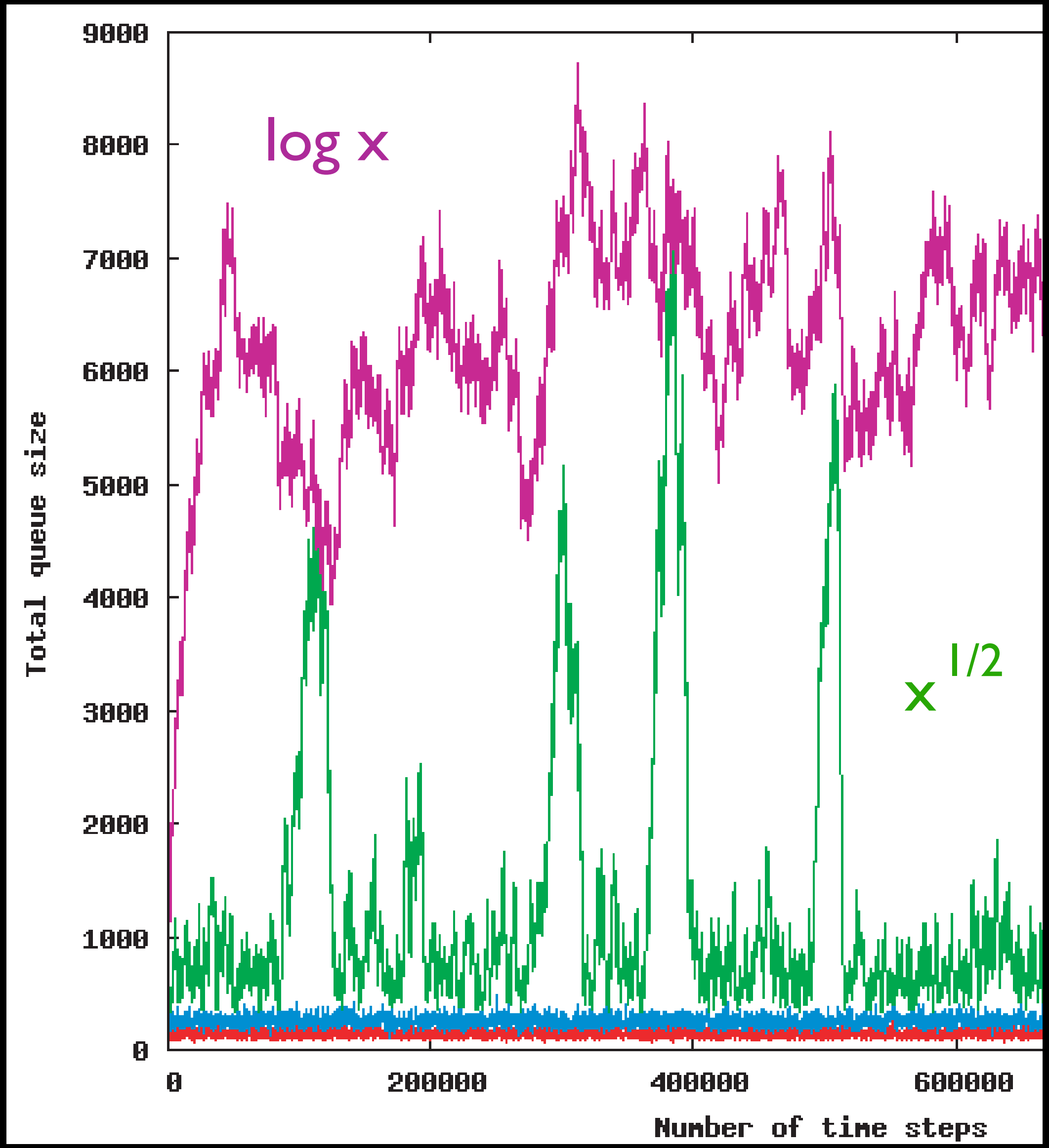

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- Hence, Queue-size = poly(n) or exp(n) depending on graph structure
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Summary of Network Adiabatic Theorem

- In summary,

Designing a high performance medium access algorithm

$\downarrow$ Step II

Sampling time-varying distribution $\pi(t)=\pi(Q(t))$ satisfying MW property [Tassiulas and Ephremides 92]

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Medium access algorithm (Queue-based CSMA) = Distributed Iterative sampling mechanism

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- Next: We generalize this framework

Generalized Network Adiabatic Theorem

- [S. and Suk 2014]* establish the following generic framework

Designing a high performance combinatorial resource allocation algorithm

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Queue-based low-complexity and distributed mechanism:
Run only one iteration of the optimization method per each time

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  Pick-and-Compare [Tassiulas 1998] is a special case using Exhaustic Search

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- We prove that throughput-optimality is guaranteed if $f(x)=(\log x)^c$ where c is the current channel state

- We also prove that log f(x) should be linear with respect to c for throughput-optimality

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My Contribution for Distributed Scheduling

Queue-based Algorithms

Network adiabatic theorem for medium access
[Shah and S.] SIGMETRICS 2009 and Annals of Applied Probability 2012

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[Shah, S. and Prasad] FOCS 2011

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Throughput optimality for rate-based CSMA
[Jiang, Shah, S. and Walrand] IEEE Transactions
on Information Theory 2010

Belief Propagation for medium access
[Yun, S. and Yi] to appear in IEEE
Transactions on Information Theory

Rate-based Algorithms

- Part II -
Message Passing in Statistical Networks

- Convergence and Correctness of Belief Propagation
  Joint work with Sungsoo Ahn (KAIST), Michael Cherktov (LANL) and Sejun Park (KAIST)

Graphical Model

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- Factor Graph for GM

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A joint distribution of n (binary) random variables $Z = [Z_i] \in \{0,1\}^n$ is called a Graphical Model (GM) if it factorizes as follows: for $z = [z_i] \in \Omega^n$,

$$\Pr[Z = z] \propto \prod_{i \in \{1,\ldots,n\}} \psi_i(z_i) \prod_{\alpha \in F} \psi_\alpha(z_\alpha),$$

where $\{\psi_i, \psi_\alpha\}$ are (given) non-negative functions, the so-called factors; F is a collection of subsets

$$F = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset 2^{\{1,2,\ldots,n\}}$$

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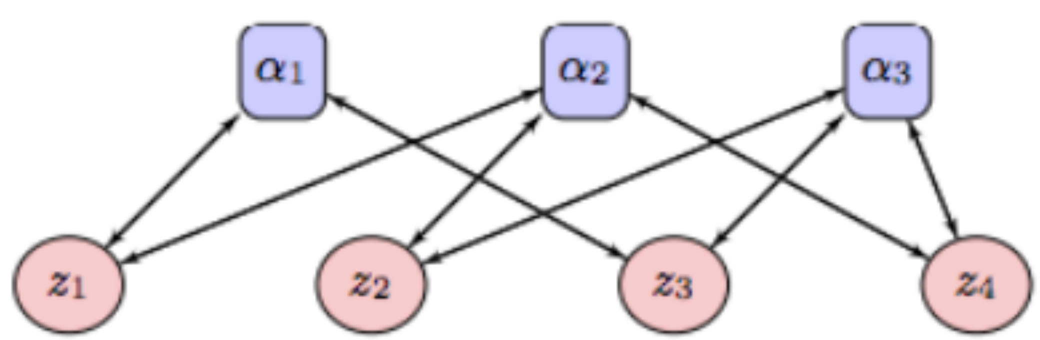

Figure 1: Factor graph for the graphical model $\Pr[z] \propto \psi_{\alpha_1}(z_1, z_3)\psi_{\alpha_2}(z_1, z_2, z_4)\psi_{\alpha_3}(z_2, z_3, z_4)$, i.e., $F = \{\alpha_1, \alpha_2, \alpha_3\}$ and $n = 4$. Each α_j selects a subset of z. For example, α_1 selects $\{z_1, z_3\}$.

Computational Challenges in Graphical Model

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Need some heuristics or approximation algorithms !

Belief Propagation for Marginal Probability

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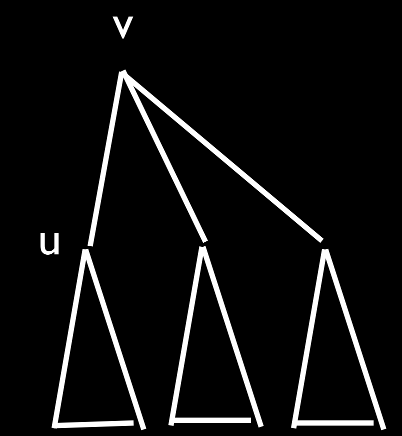

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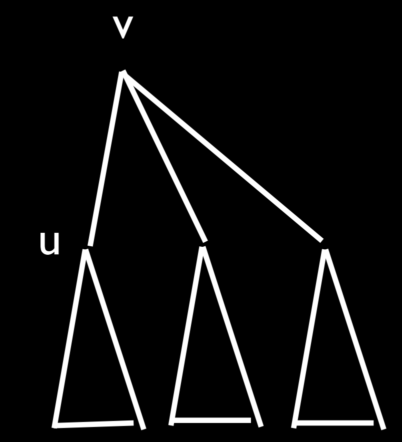

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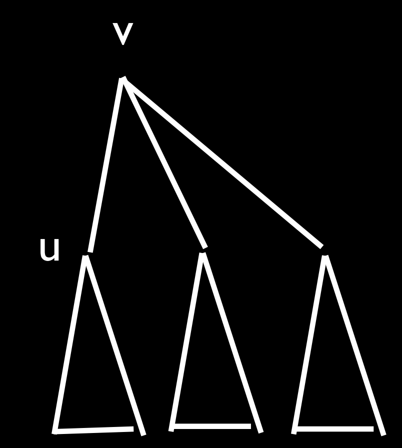

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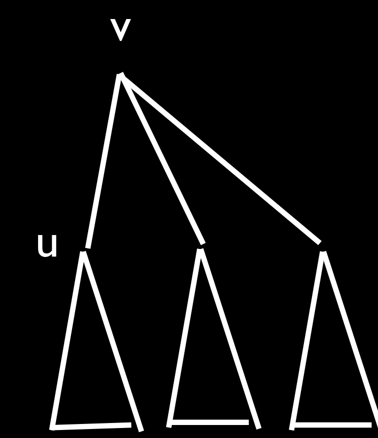

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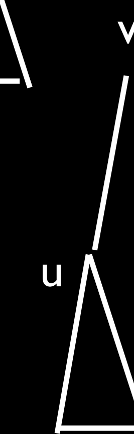

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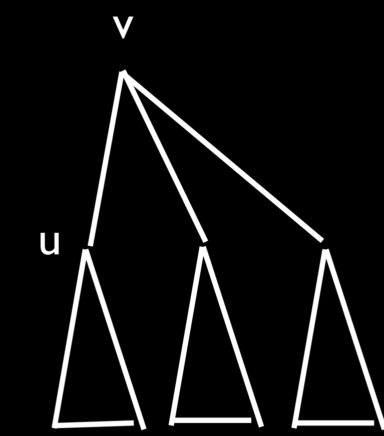

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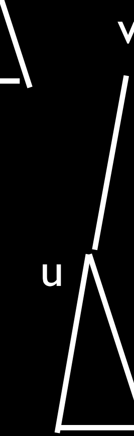

$$M_{u\to v} = \frac{1}{1+\prod_{w\in N(u)/v} M_{w\to u}}$$

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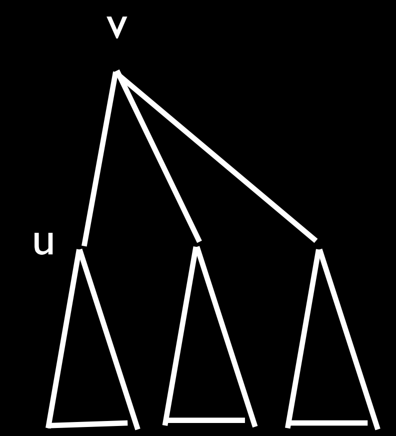

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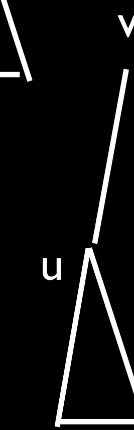

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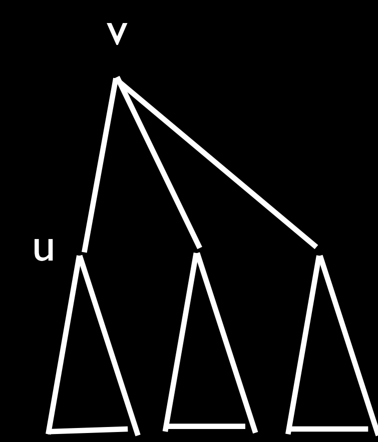

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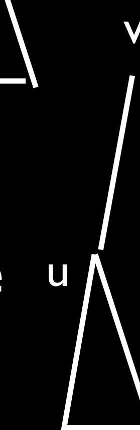

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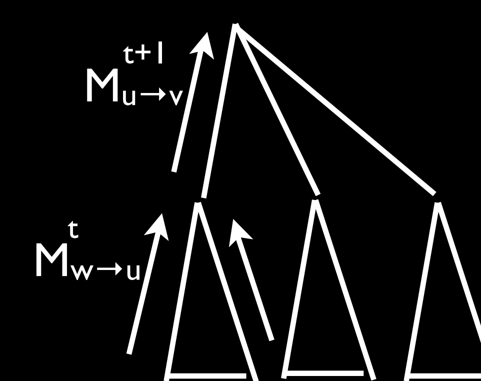

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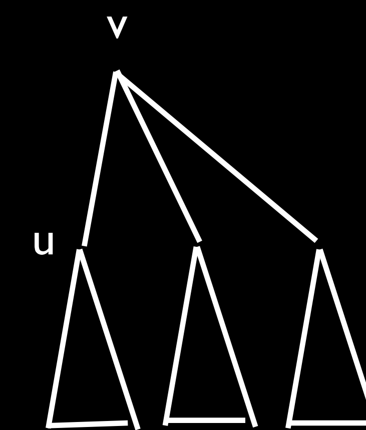

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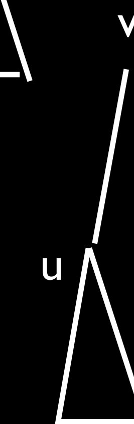

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It is dynamic programming !

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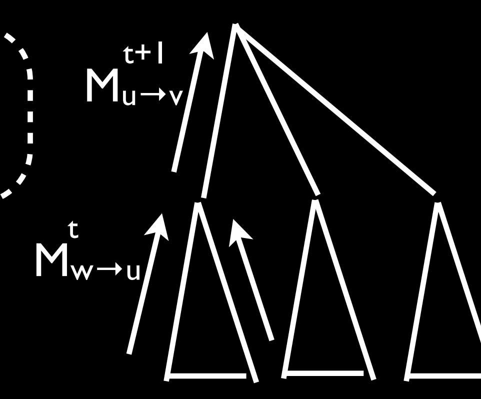

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- First, M^t converges to a fixed point of f_{BP} (and how fast)?
  - A fixed point always exists due to the Brouwer fixed point theorem, but BP often diverges

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  - Convergence conditions : [Weiss 00] [Tatikonda and Jordan 02] [Heskes 04] [Ihler et al. 05]
  - However, they are very sensitive w.r.t potential functions and the underlying graph structure

- Second, a fixed (i.e. convergent) point of f_{BP} is good ?
  - Several efforts : [Wainwright et al. 03] [Heskes 04] [Yedidia et al. 04] [Chertkov et al. 06]
  - However, they provide different 'views' instead of simple conditions

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Any algorithm should take $\Omega(n)$ iterations

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[S. et al. 2011]† Fixed points of f_{BP} are good if girth = $\Omega(\log n)$

† Computing Indep. Sets using Bethe Approximation [Shin et al.] SIDMA 2011

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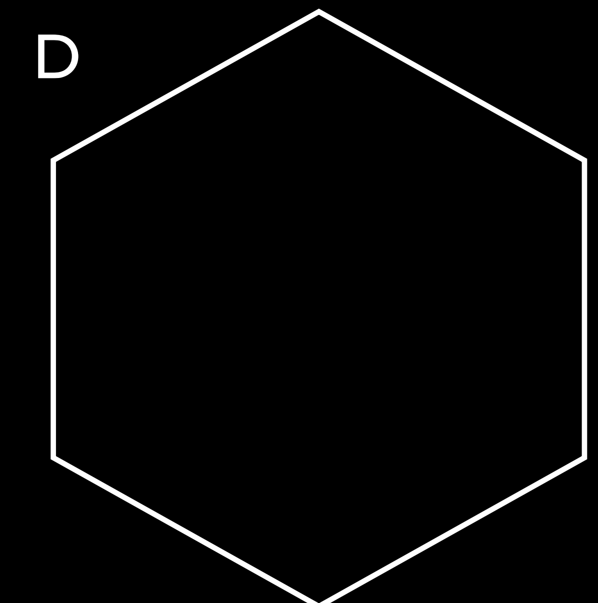
Proof Strategy : How to find a fixed point of SBP

- Equivalent to find a zero-gradient point of F_{Bethe} [Yedidia et al. 04]
  - $F_{Bethe} : D \rightarrow R$ is called the Bethe free energy function [Bethe 35]
  - The underlying domain D is a polytope
  - It is not clear whether it is easy to find (or PLS-hard) since it is non-convex



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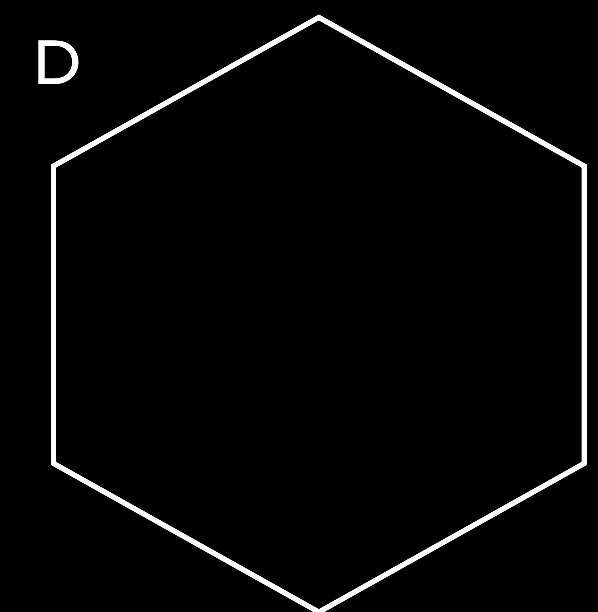

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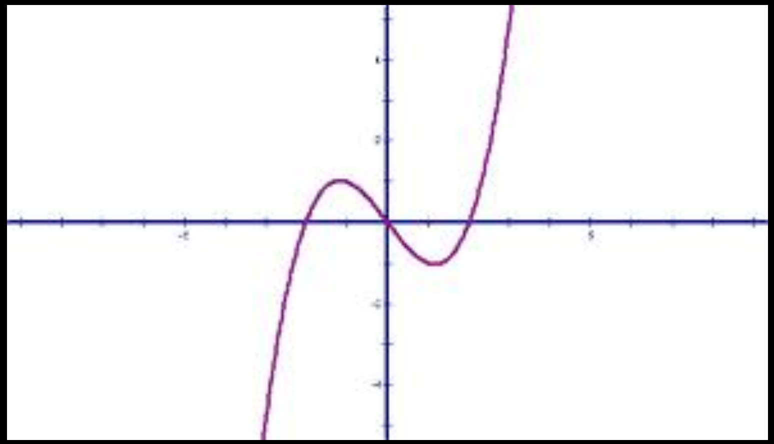
- Natural attempt : Gradient-descent algorithm

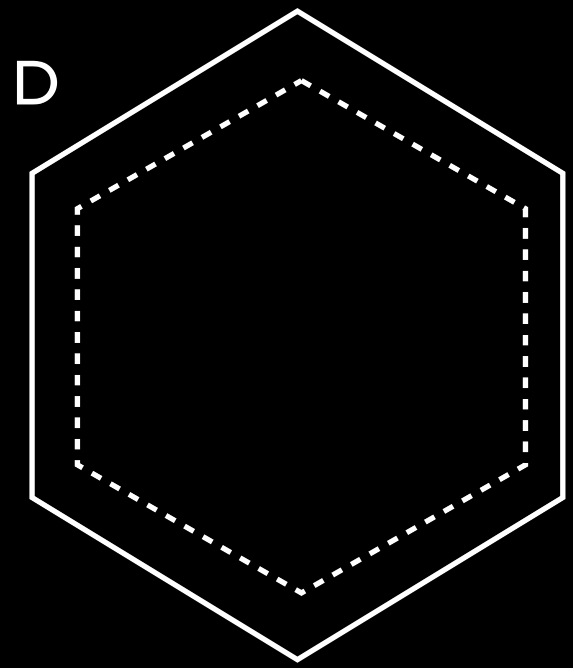
$$x^{t+1} = x^t + \alpha \nabla F_{Bethe}(x^t)$$ for some (step-size) $\alpha>0$

  - Does it find to a zero-gradient point? If not, why?

Proof Strategy : How to find a fixed point of LBP

- When does gradient-descent algorithm work for general F ?
  - Its domain is unbounded and $|F|, |\nabla F|, |\nabla^2 F|$ are bounded $\rightarrow$ Possible to choose α





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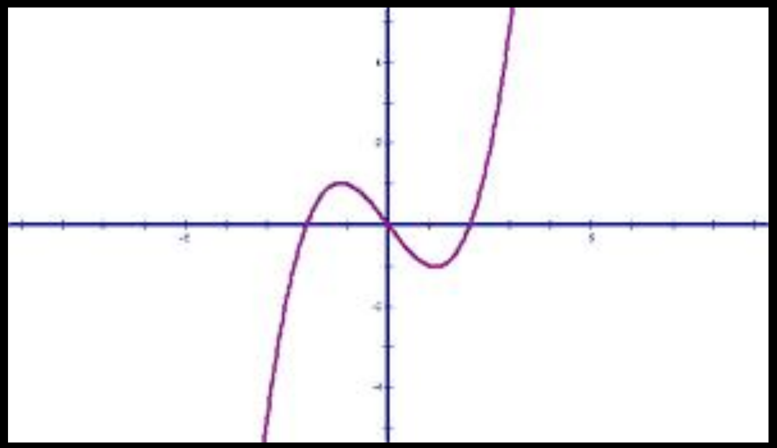
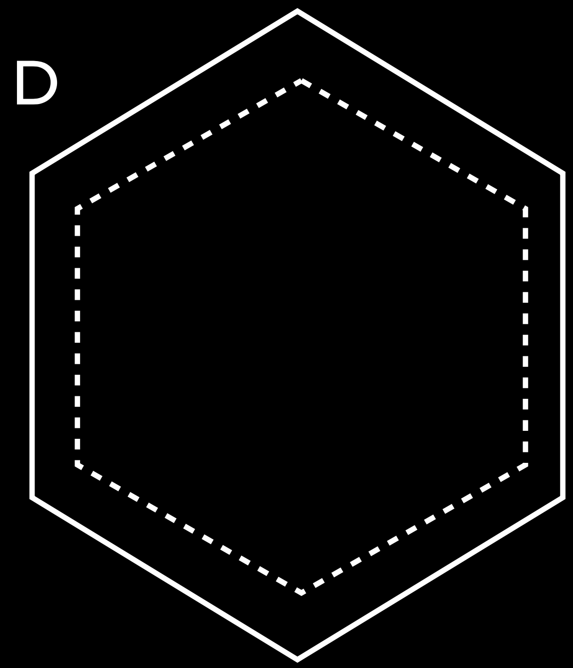
- Two issues for $x^{t+1} = x^t + \alpha \nabla F_{Bethe}(x^t)$
  - Domain D is bounded i.e. a projection may be required
  - Derivatives are unbounded (close to boundary of D)





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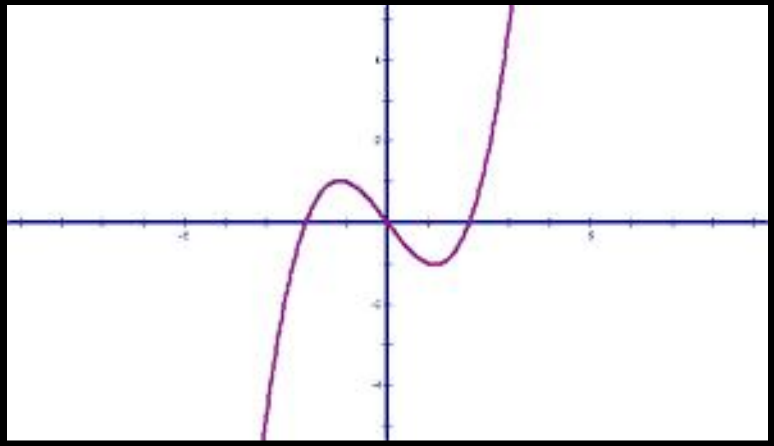

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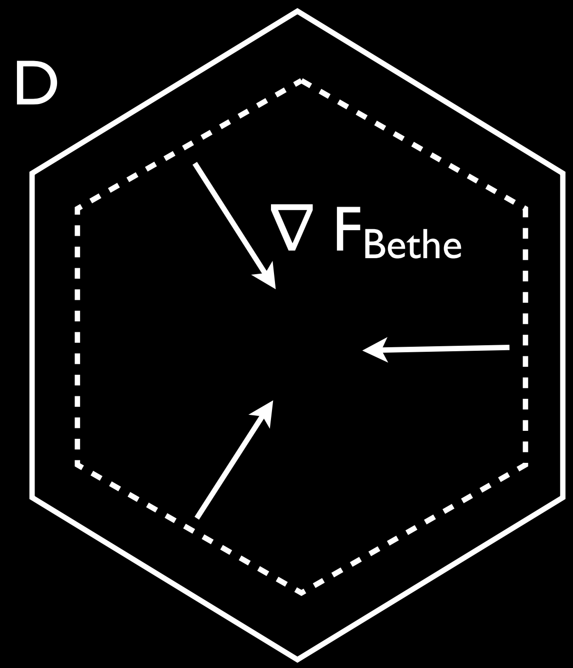

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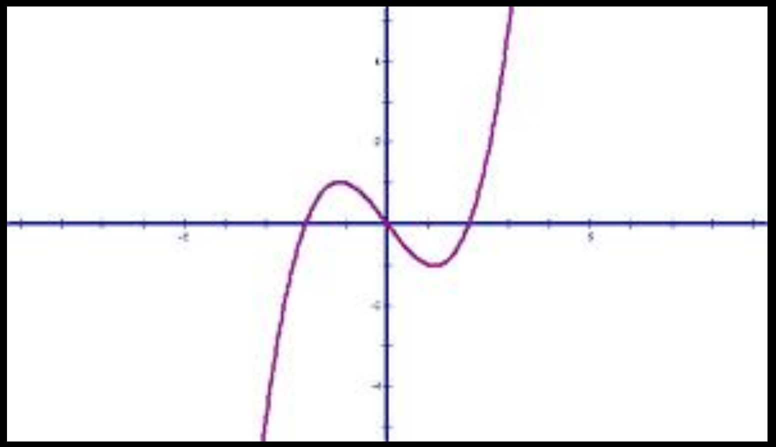

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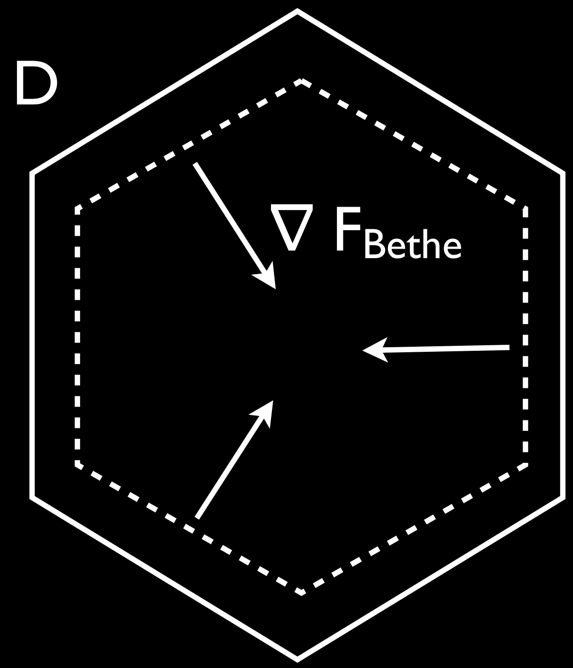

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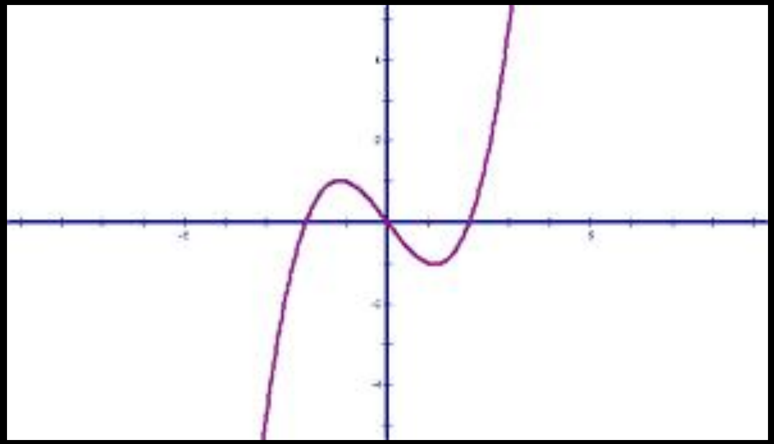

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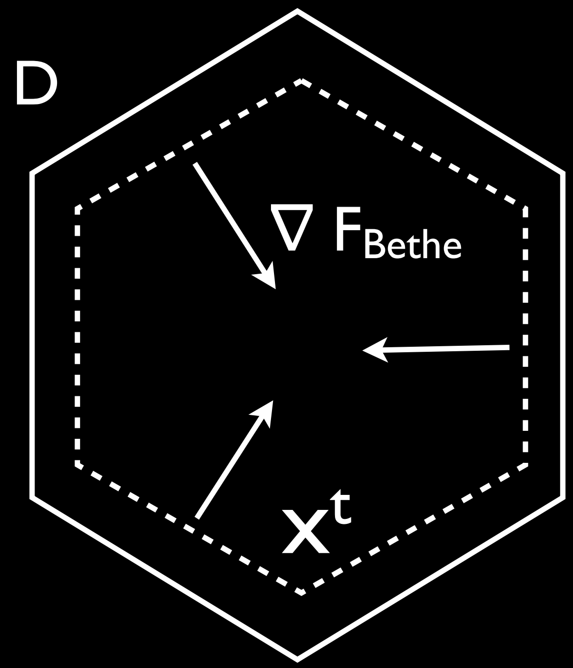

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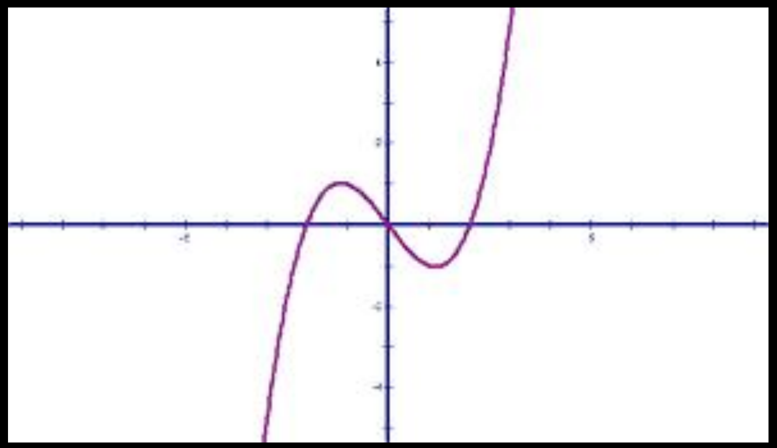

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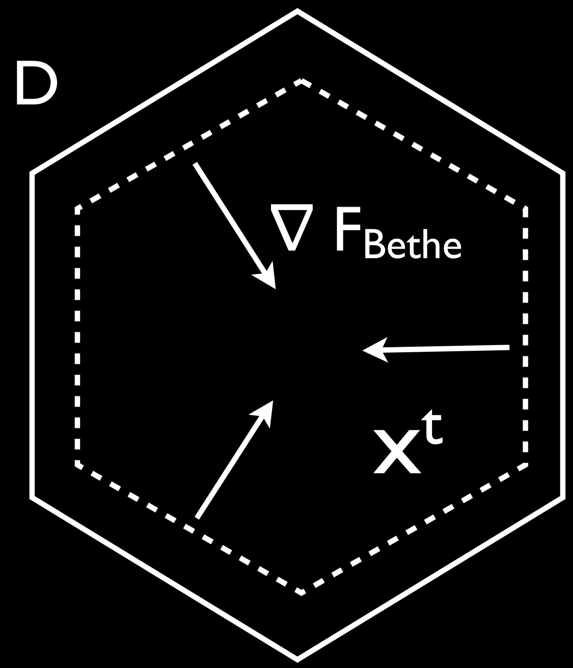

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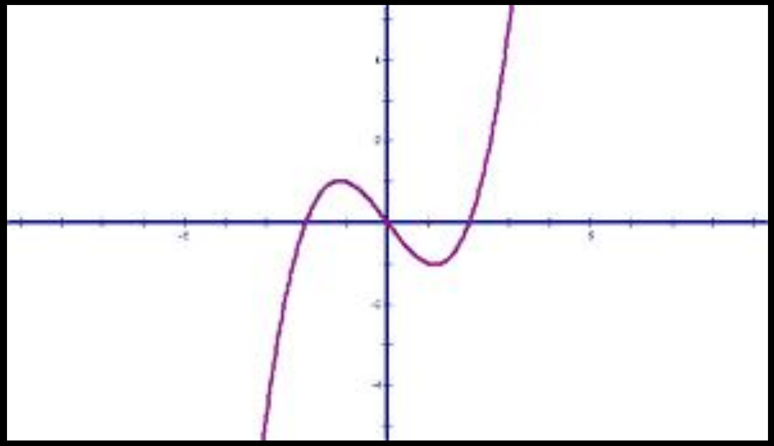

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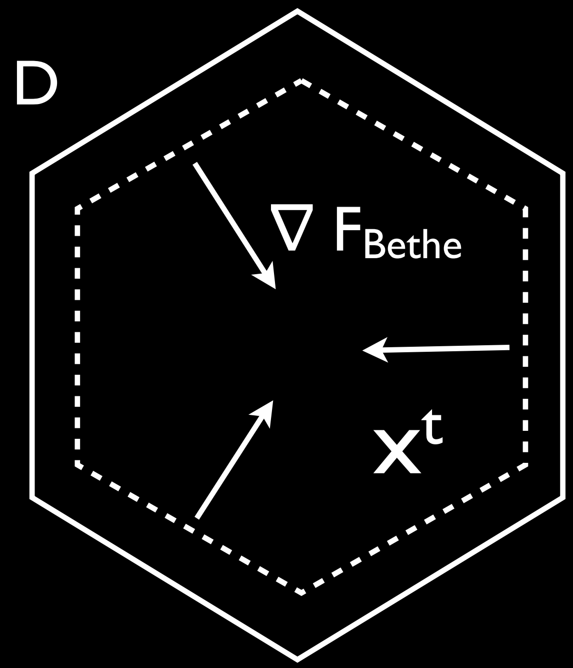

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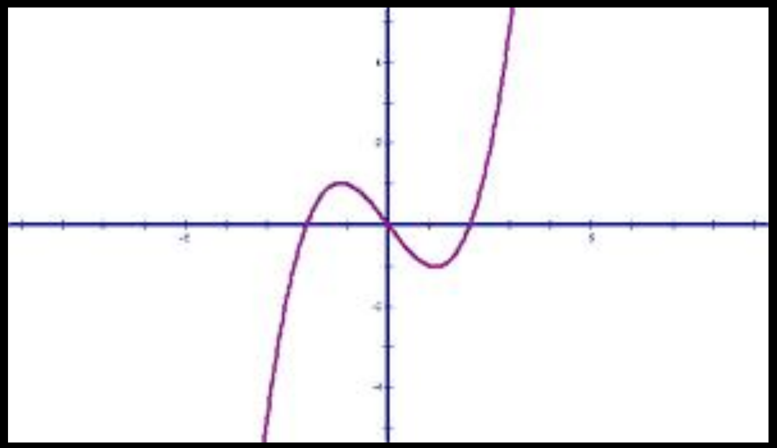

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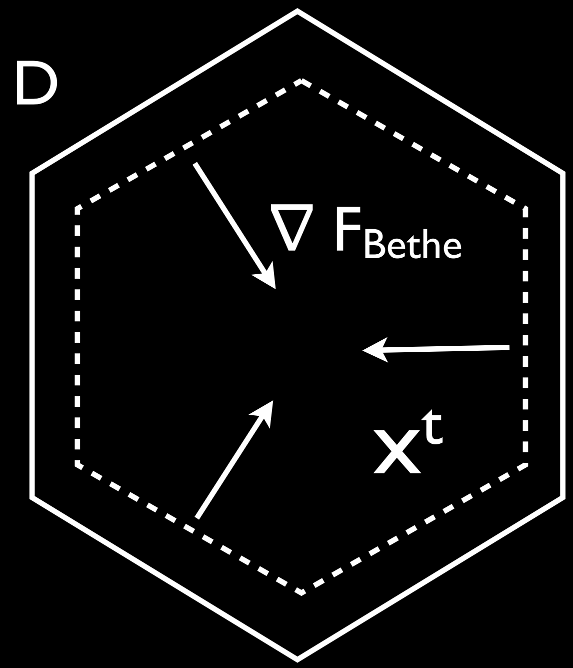

We found a function G such that this property holds for G and one-to-one correspondence between zero-gradients of G and F_{Bethe}

How to understand Max-product BP ?

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  - C1. LP has a unique and integral solution
  - C2. Each variable is associated to at most two factors
  - C3. *For every factor ψ_α, every $x_\alpha \in \{0,1\}^{|\alpha|}$ with $\psi_\alpha(x_\alpha) = 1$, and every $i \in \alpha$ with $x_i \neq x_i^*$, there exists $\gamma \subset \alpha$ such that*

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- Why BP can be better than simplex or interior-point methods ?
  - BP is easy to parallelize and implement in a distributed & parallel programming model

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Examples of LP solvable by BP

Shortest Path

$$\begin{array}{ll} \text{minimize} & w \cdot x \\ \text{subject to} & \displaystyle\sum_{e \in \delta^o(v)} x_e - \sum_{e \in \delta^i(v)} x_e \\ & = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V \\ & x = [x_e] \in [0,1]^{|E|}. \end{array}$$

Minimum Weight (Perfect) Matching

$$\begin{array}{ll} \text{minimize} & w \cdot x \\ \text{subject to} & \displaystyle\sum_{e \in \delta(v)} x_e = 1, \quad \forall v \in V \\ & x = [x_e] \in [0,1]^{|E|} \end{array}$$

Vertex Cover

$$\begin{array}{ll} \text{minimize} & b \cdot y \\ \text{subject to} & y_u + y_v \geq 1, (u,v) \in E \\ & y = [y_v] \in [0,1]^{|V|}. \end{array}$$

Cycle Packing

$$\begin{array}{ll} \text{maximize} & w \cdot x \\ \text{subject to} & \displaystyle\sum_{e \in \delta(v)} x_e = 2 y_v \\ & x = [x_e] \in [0,1]^{|E|}, y = [y_v] \in [0,1]^{|V|}. \end{array}$$

Traveling Salesman Problem

$$\begin{array}{ll} \text{minimize} & w \cdot x \\ \text{subject to} & \displaystyle\sum_{e \in \delta(v)} x_e = 2 \\ & x = [x_e] \in [0,1]^{|E|} \end{array}$$

Network Flow

$$\begin{array}{ll} \text{minimize} & w \cdot x \\ \text{subject to} & \displaystyle\sum_{e \in \delta^o(v)} x_e - \sum_{e \in \delta^i(v)} x_e = d_v, \ \forall v \in V \\ & x = [x_e] \in \mathbb{R}_+^{|E|} \end{array}$$

My Contribution for Belief Propagation

- Sum-product BP
  - Polynomial-time algorithm for computing a BP fixed-point
    [S.] IEEE Transactions on Information Theory 2014
  - Large-girth condition for correctness of BP
    [S. et al.] SIAM Journal on Discrete Mathematics 2011
- Max-product BP
  - BP solves the LP for minimum weight matching using odd cycle constraints
    [S., Gelfand and Chertkov] NIPS 2013
  - Generic necessary condition so that BP solves LP
    [Gelfand, Chertkov and S.] ISIT 2013
  - Generic sufficient condition so that BP solves LP
    [Park and S.] UAI 2015
  - BP solves the IP for minimum weight matching
    [Ahn, Chertkov, Park and S.] submitted

Summary

Part I

Scheduling algorithms for
communication networks
(e.g. wireless communication)

Part II

Efficient algorithms for
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Scheduling algorithms for
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Message Passing Algorithms

Part II

Efficient algorithms for
Statistical Inference
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Why Message Passing Algorithms ?

- They are crucial for numerous fields in engineering and social science

  - Building blocks for communication (Internet) networks
    e.g. medium access, packet switching

  - Efficient estimation tools for statistical (Bayesian) networks
    e.g. variational (or cavity) method, Markov chain monte carlo

  - Faithful behavioral models for societal systems
    e.g. markets, auctions, recommendation systems

My Research = Principles of Local Rules for Networked Systems