Faster Greedy MAP Inference for Determinantal Point Processes

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Determinantal Point Processes (DPPs)

Given a ground set $\mathcal{Y} = \{1, \dots, d\}$ and positive definite matrix $L \in \mathbb{R}^{d \times d}$,

$$\Pr(X) \propto \det(L_X) \quad \text{for} \quad X \subseteq \mathcal{Y},$$

where L_X is a submatrix of L indexed by items of X.

- DPPs are probabilistic models capturing both diversity and item quality of subsets.
- Most inference tasks (including normalization, marginalization, conditioning and sampling) can be done in $O(d^3)$.
- However, MAP inference is known as **NP-hard** problem, that is,

$$\arg\max_{X \subseteq \mathcal{Y}} \det L_X.$$

- The MAP inference of DPP has been used for many machine learning applications, e.g., text/video summarization, change-point detection, and informative image search.

Our Contribution: Faster MAP Inference of DPP

Since $\log\det$ is a submodular function, greedy algorithms for approximating MAP of DPP have been of typical choice.

- A naïve greedy algorithm requires $O(d^5)$ operations.

algorithm	complexity	remarks
[Minoux, 1978]	$O(d^5)$	accelerated version of a naïve greedy algorithm
[Buchbinder et al., 2015]	$O(d^4)$	symmetric greedy algorithm
[Gillenwater et al., 2012]	$O(d^4)$	multilinear softmax extension

We propose faster greedy algorithms requiring $O(d^3)$ operations.

First Ideas: Talyor Expansion

Greedy algorithms require computing the following marginal gains:

$$\log\det L_{X \cup \{i\}} - \log\det L_X$$

For their efficient computations, our key ideas are:

1. **First-order Taylor expansion for Log-determinant**

$$\log\det L_{X \cup \{i\}} - \log\det \overline{L}_X \;\approx\; \left\langle \overline{L}_X^{-1}, L_{X \cup \{i\}} - \overline{L}_X \right\rangle.$$

   - $\overline{L}_X$ is the average of $L_{X \cup \{i\}}$ for $i \in \mathcal{Y} \setminus X$.
   - $L_{X \cup \{i\}}$ and $\overline{L}_X$ differ only single column and row.
   - Single column of $\overline{L}_X^{-1}$ is computed by a linear solver, e.g., conjugate gradient descent.

2. **Partitioning**

   - For much tighter approximation, we divide $\mathcal{Y} \setminus X$ into p partitions so that

$$\|L_{X \cup \{i\}} - \overline{L}_X\|_F \gg \|L_{X \cup \{i\}} - \overline{L}_X^{(j)}\|_F,$$

   where i is in the partition $j \in \{1, \dots, p\}$.
   - To compute the marginal gains, we need to calculate extra term $(*)$:

$$\log\det L_{X \cup \{i\}} - \log\det L_X$$
$$\approx \underbrace{\left\langle (\overline{L}_X^{(j)})^{-1}, L_{X \cup \{i\}} - \overline{L}_X^{(j)} \right\rangle}_{\text{can compute by a linear solver}} + \underbrace{\left(\log\det \overline{L}_X^{(j)} - \log\det L_X \right)}_{(*)}.$$

   - $(*)$ is also computable by a linear solver under Schur complement.

The overall complexity becomes $O(d^3)$ because we choose $p = O(1)$ and

- In each greedy step, a linear solver can be used to compute both Taylor approximation and $(*)$, thus $O(p \times d^2)$ operations are required.
- The total number of greedy steps is at most d.

Second Ideas: Batch Strategy

We consider adding k-batch subset (instead of single element)

$$X \leftarrow X \cup I \quad \text{for some } |I| = k > 1$$

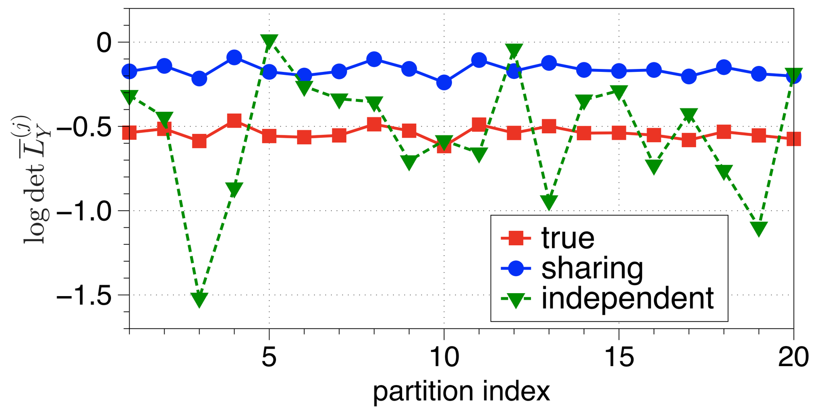
so that the number of greedy steps can be reduced at most k times.

1. **Sampling random batches**
   - For the optimal k-batch, one has to investigate $\approx \binom{d}{k}$ subsets.
   - This is expensive. Instead, we randomly sample batches and add the best of them to the current set.

2. **Log-determinant approximation under sharing randomness**
   - For k-batch strategy, one can compute the extra term $(*)$, i.e., $\log\det \overline{L}_X^{(j)} - \log\det L_X$, by running a linear solver k times.
   - Alternatively, we suggest estimating all log-determinants $\log\det \overline{L}_X^{(j)}$ by running a log-determinant approximation scheme (LDAS) [Han et al., 2015], but only once.

method	complexity	number of calls	objective
linear solver	$O(d^2)$	k	$\log\det \overline{L}_X^{(j)} - \log\det L_X$
LDAS	$O(d^2)$	1	$\log\det \overline{L}_X^{(j)}$

   - LDAS approximates $\log\det \overline{L}_X^{(j)}$ using independent random vectors. We suggest to run LDAS using the same random vectors for estimating all $\log\det \overline{L}_X^{(j)}$.



Observe that running LDAS's under sharing random vectors is better for comparing $\log\det \overline{L}_X^{(j)}$, i.e., marginal gains.

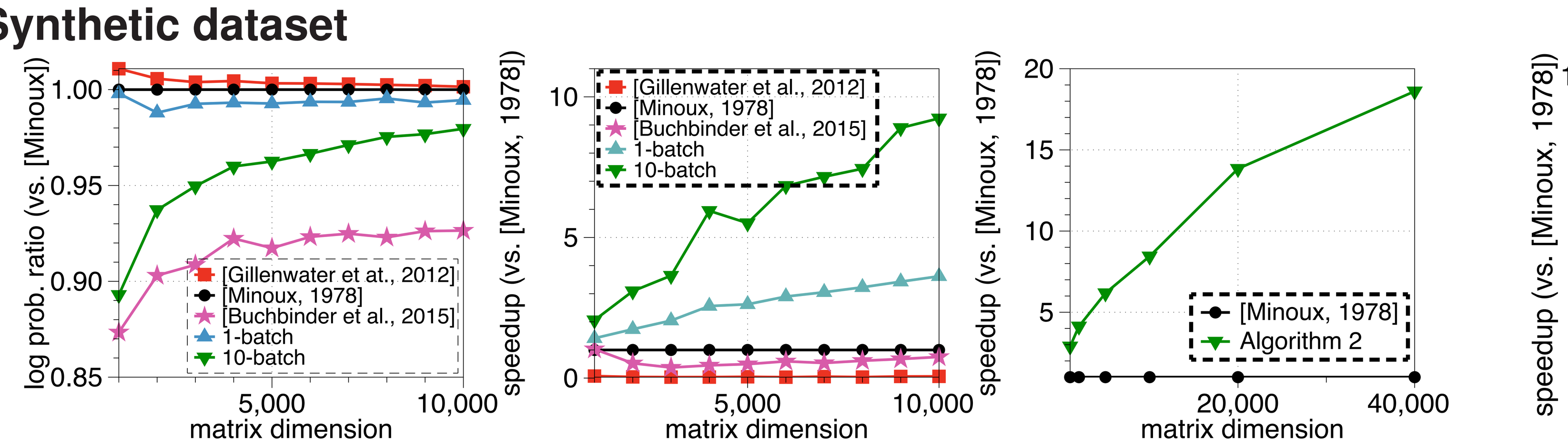
   - We provide the following error bound of LDAS under sharing random vectors, where $A = \overline{L}_X^{(j)}$ and $B = \overline{L}_X^{(j')}$.

> **Theorem** (Han, Prabhanjan, Park and Shin, 2017). *Suppose A, B are positive definite matrices whose eigenvalues are in $[\delta, 1-\delta]$ for $\delta > 0$. Let Γ_A, Γ_B be the estimations of $\log\det A$, $\log\det B$ by LDAS using the same m random vectors for both. Then, it holds that*
>
> $$\mathbf{Var}\left[\Gamma_A - \Gamma_B\right] \le \frac{32 M^2 \rho^2 (\rho+1)^2}{m (\rho-1)^6 (1-2\delta)^2} \|A - B\|_F^2$$
>
> *where $M = 5\log(2/\delta)$ and $\rho = 1 + \frac{2}{\sqrt{2/\delta - 1} - 1}$.*

On the other hand, the variance of LDAS under independent random vectors depends on $\|A\|_F^2 + \|B\|_F^2$ which is significantly larger than $\|A - B\|_F^2$ in our case.

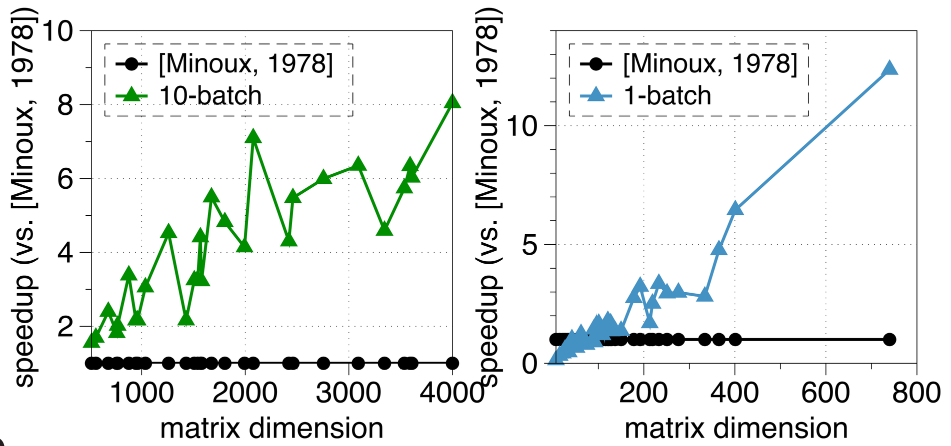
Experiments

Synthetic dataset



- Accuracy is measured by log-probability ratio of a respective algorithm to the standard (but accelerated) greedy algorithm [Minoux, 1978].
- Two versions of our algorithms: 1-batch and 10-batch with 50 batch samples.
- 1-batch achieves 0.03% and 10-batch achieves 0.3% loss on accuracy.
- 10-batch runs up-to 18 times faster than [Minoux, 1978].

Real dataset: Text and video summarizations



- The number of selected items in video summarization is small. In this case, 1-batch shows better performance than 10-batch.
- For both text and video summarization task, our algorithms run $8 \sim 10$ times faster than [Minoux, 1978] for large instances.
- Our video summaries often have higher F-score than [Minoux, 1978].

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