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A Graphical Transformation for Belief Propagation: Maximum Weight Matchings and Odd-Sized Cycles

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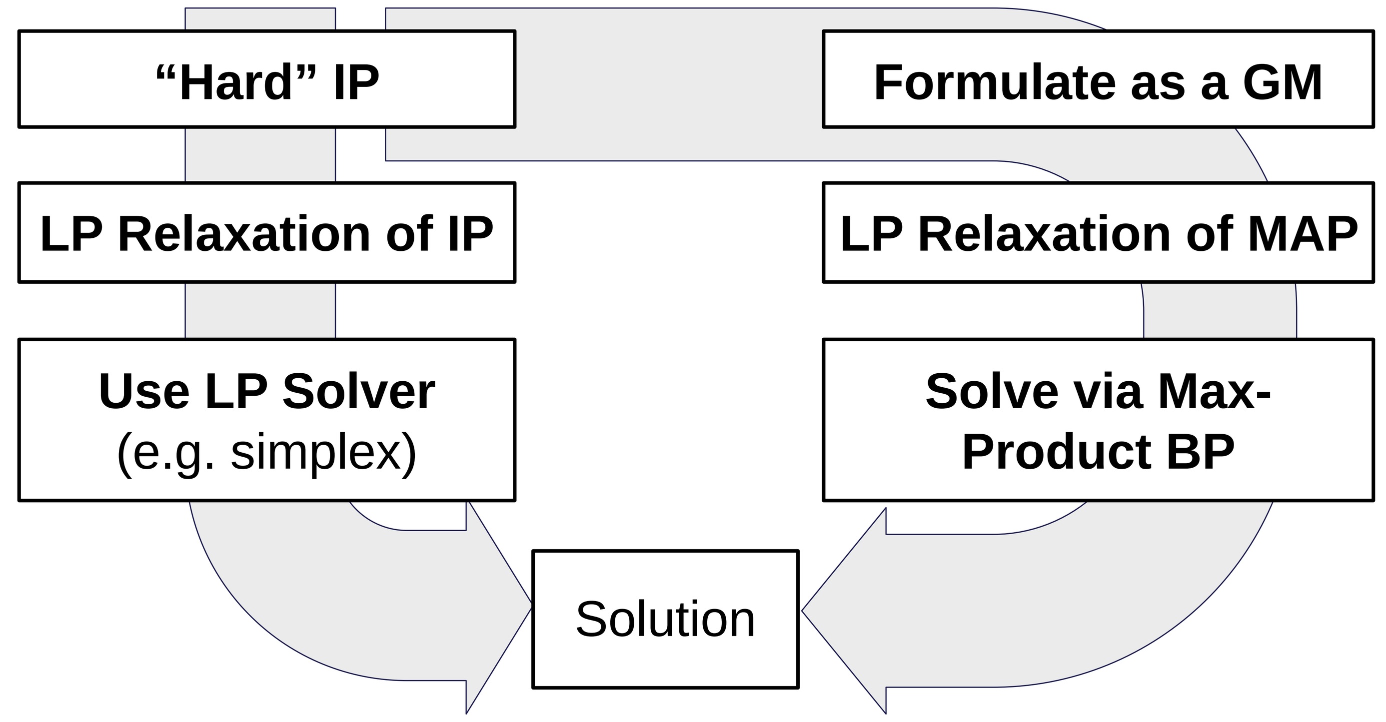
Summary

- Max-Product Belief Propagation (BP) is a successful heuristic for finding the MAP assignment in probabilistic graphical models (GM)
- BP was recently shown to be convergent & correct in loopy GMs corresponding to classic combinatorial optimization problems
- A necessary condition for BP to succeed in such problems is that the LP relaxation of the MAP assignment problem be tight
- Tightness is ***not*** a sufficient condition however, and we explore why BP fails even when the LP is tight
- We focus on the classic Max Weight Matching problem and propose a novel graphical transformation that forces BP to converge when the LP relaxation of the matching problem is tight
- This is an important step towards designing BP algorithms guaranteed to recover the LP solution

Max-Product BP as an LP Solver



Max Weight Matching Problem

Matching is a subset of *E* w/ at most 1 edge adjacent to each *V*

Integer Program:

$$\max \sum_{e\in E} w_e x_e$$
$$\text{s.t. } \sum_{e\in\delta(i)} x_e \le 1, \ \forall i \in V$$
$$x_e \in \{0,1\}$$

LP–Relaxation:

$$\max \sum_{e\in E} w_e x_e$$
$$\text{s.t. } \sum_{e\in\delta(i)} x_e \le 1, \ \forall i \in V$$
$$x_e \in [0,1]$$
$$\sum_{e\in E(S)} x_e \le \frac{|S|-1}{2}, \ \forall S \in \mathcal{S}$$

Edmonds' Blossom Constraints

- LP-Relaxation is **tight** when cut constraints for all odd-sized subsets, *S*, have been added
- Chandrasekaran et al. 2012, showed that # of cuts, *S*, is polynomial
- We will consider sets $\mathcal{C}$ of non-intersecting, odd-sized cycles

Max-Product for Matching

Graphical Model Formulation (GM-0):

$$\max_x \prod_{e\in E} \exp(w_e x_e) \prod_{i\in V} \psi_i(x_i) \prod_{C\in\mathcal{C}} \psi_C(x_C)$$

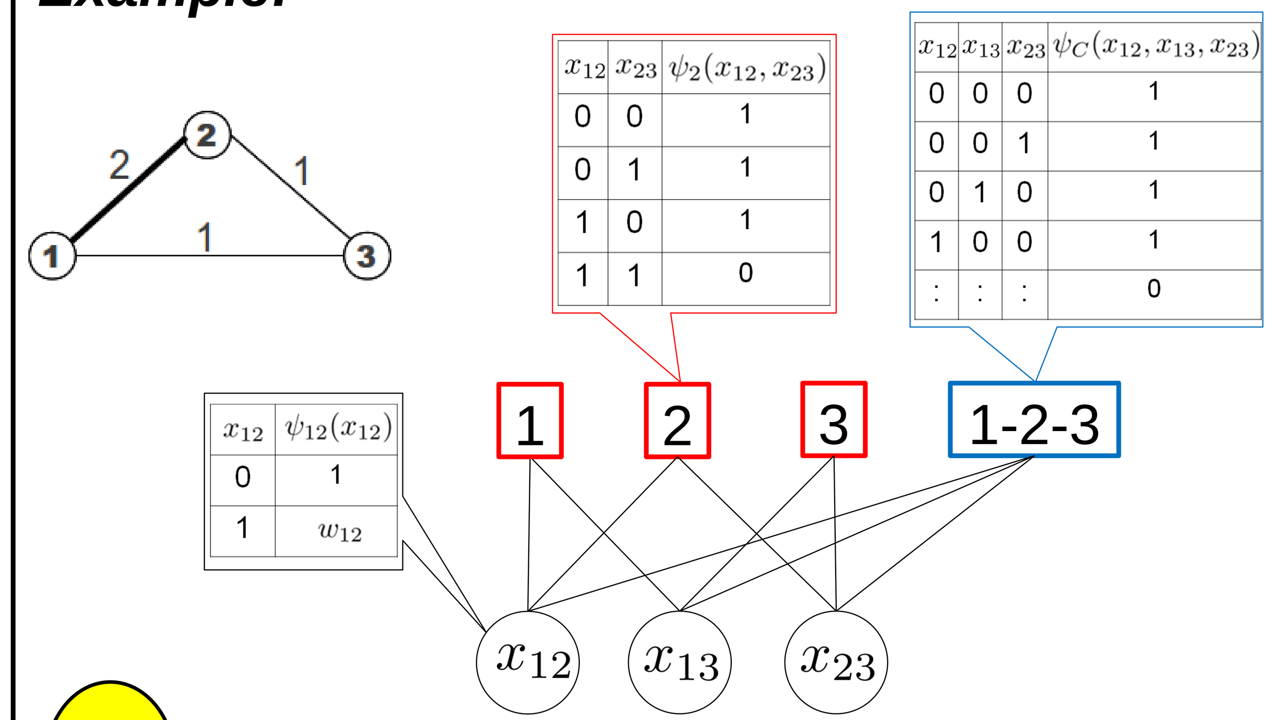
$$\psi_i(x_i) = \begin{cases} 1 & \text{if } \sum_{e\in\delta(i)} x_e \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_C(x_C) = \begin{cases} 1 & \text{if } \sum_{e\in E(C)} x_e \le \frac{|C|-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Corresponding LP–Relaxation (C-LP):

$$\max \sum_{e\in E} w_e x_e$$
$$\text{s.t. } \sum_{e\in\delta(i)} x_e \le 1, \ \forall i \in V \quad x_e \in [0,1]$$
$$\sum_{e\in E(C)} x_e \le \frac{|C|-1}{2}, \ \forall C \in \mathcal{C} \longleftarrow \text{non-intersecting, odd-sized cycles}$$

Example:



Max-Product BP doesn't converge even though corresponding LP, C-LP, is tight!

Iteration Number

		0	1	2	3	4	5
Variable-to-Factor Messages:	x_12→1		0	-2	0	-2	0
	x_12→2		0	-2	0	-2	0
	x_13→1		0	-1	2	-1	2
	x_13→3		0	-1	3	-1	3
	x_23→2		0	-1	2	-1	2
	x_23→3		0	-1	3	-1	3
	x_12→1-2-3		0	-2	0	-2	0
	x_13→1-2-3		0	-1	2	-1	2
	x_23→1-2-3		0	-1	2	-1	2
Factor-to-Variable Messages:	1→x_12	0	0	1	0	1	0
	2→x_12	0	0	1	0	1	0
	1→x_13	0	0	2	0	2	0
	3→x_13	0	0	1	0	1	0
	2→x_23	0	0	2	0	2	0
	3→x_23	0	0	1	0	1	0
	1-2-3→x_12	0	0	1	0	1	0
	1-2-3→x_13	0	0	2	0	2	0
	1-2-3→x_23	0	0	2	0	2	0
Edge Beliefs:	x_12	-2	-2	1	-2	1	-2
	x_13	-1	-1	4	-1	4	-1
	x_23	-1	-1	4	-1	4	-1

Graphical Transformation

- Sanghavi, Malioutov & Willsky showed that BP is convergent and correct if C-LP with $\mathcal{C} = \emptyset$ is tight (and unique)
- We "fix" BP in situations where C-LP is tight, but $\mathcal{C} \neq \emptyset$

New **Graphical Model Formulation (GM-1):**

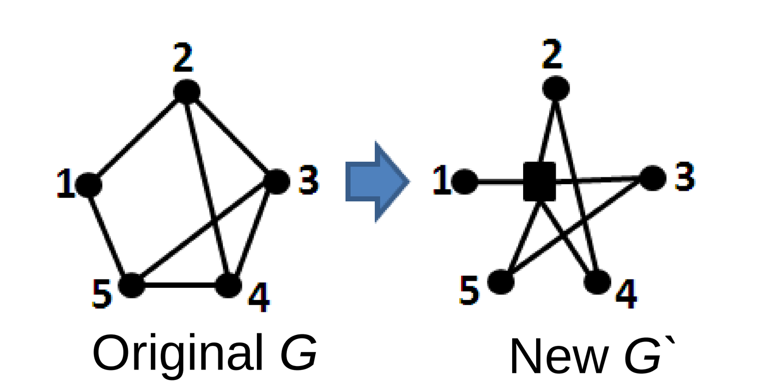
$$\max_y \prod_{e\in E'} \exp(w'_e y_e) \prod_{i\in V'} \psi_i(y_i) \prod_{C\in\mathcal{C}} \psi_C(y_C)$$

$$\psi_C(y_C) = \begin{cases} 0 & \text{if } \sum_{e\in\delta(i_C)} y_e > |C|-1 \\ 0 & \text{if } \sum_{j\in V(C)} (-1)^{d_C(j,e)} y_{i_C,j} \notin \{0,2\} \text{ for an } e \in E(C) \\ 1 & \text{otherwise} \end{cases}$$

$$w'_e = \begin{cases} \frac{1}{2} \sum_{\tilde{e}\in E(C)} (-1)^{d_C(j,\tilde{e})} w_{\tilde{e}} & \text{if } e = (i_C, j) \text{for some } C \in \mathcal{C} \\ w_e & \text{otherwise} \end{cases}$$

Example:

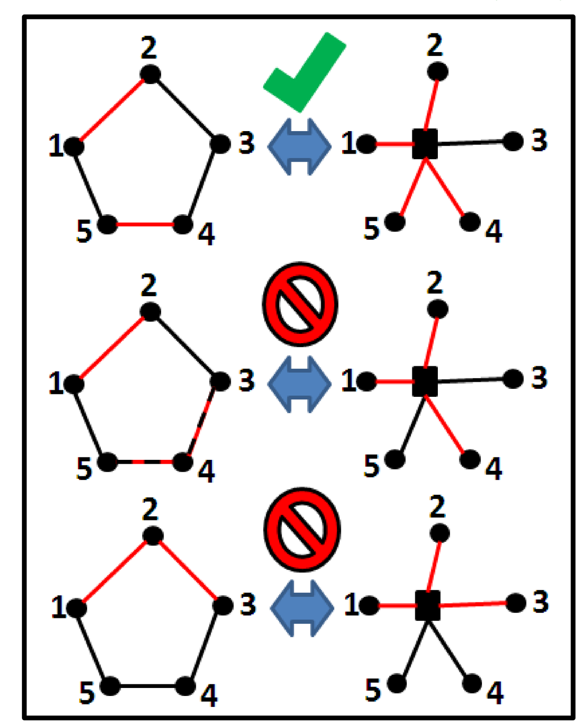
Collapse cycle *C=(1,2,3,4,5)*



<u>*New* edge weights</u>:

$$w_{1C} = \frac{1}{2}(w_{12} - w_{23} + w_{34} - w_{45} + w_{15})$$
$$w_{2C} = \frac{1}{2}(w_{12} + w_{23} - w_{34} + w_{45} - w_{15})$$
$$\vdots$$

1-to-1 mapping under $\psi_C(y_C)$



***Main Result*:**

If the solution of C-LP is integral and unique, then BP on the *new* GM converges to the corresponding MAP assignment $y^\star$ and the max weight matching $x^\star$ is recovered as:

$$x^\star_e = \begin{cases} \frac{1}{2} \sum_{j\in V(C)} (-1)^{d_C(j,e)} y^\star_{i_C,j} & \text{if } e = (i_C, j) \text{ for some } C \in \mathcal{C} \\ y^\star_e & \text{otherwise} \end{cases}$$

- Proof utilizes the computation tree technique to establish a contradiction
- Each variable is adjacent to at most 2 factor nodes in the *new* GM
  - This '*degree-two*' condition is crucial to proof of convergence & correctness

Cutting-Plane BP Algorithm:

1. Initialize $\mathcal{C} = \emptyset$
2. Run max-product BP for *T* iterations
3. For each edge *e*: $y_e = \begin{cases} 1 & \text{if } n_e^T[1] > n_e^T[0] \text{ and } n_e^{T-1}[1] > n_e^{T-1}[0] \\ 0 & \text{if } n_e^T[1] < n_e^T[0] \text{ and } n_e^{T-1}[1] < n_e^{T-1}[0] \\ 1/2 & \text{otherwise} \end{cases}$
4. Recover x from y and terminate if $x \notin \{0, 1/2, 1\}$
5. **if** $x_e \in \{0,1\}$ **then** stop and output solution;
   else if can find a non-intersecting odd-sized cycle C **then** add it to $\mathcal{C}$ and go to step 2.
   else terminate

Variable-to-Factor Message:

$$m^{t+1}_{e\to\alpha}(x_e) = \psi_e(x_e) \prod_{\alpha'\in N(e)\setminus\alpha} m^t_{\alpha'\to e}(x_e)$$

Factor-to-Variable Message:

$$m^{t+1}_{\alpha\to e}(x_e) = \max_{x_\alpha \setminus x_e} \left[\psi_\alpha(x_\alpha) \prod_{e'\in N(\alpha)\setminus e} m^t_{e'\to\alpha}(x_{e'}) \right]$$

Edge Belief: $\eta^t_e(x_e) = \psi_e(x_e) \prod_{\alpha\in N(e)} m^t_{\alpha\to e}(x_e)$