Incentivizing Strategic Users for Social Diffusion: Quantity or Quality?*

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***Abstract*—We consider a problem of how to effectively diffuse a new product over social networks by incentivizing selfish users. Traditionally, this problem has been studied in the form of influence maximization via seeding, where most prior work assumes that seeded users unconditionally and immediately start by adopting the new product and they stay at the new product throughout their lifetime. However, in practice, seeded users often adjust the degree of their willingness to diffuse, depending on how much incentive is given. To address such diffusion willingness, we propose a new incentive model and characterize the speed of diffusion as the value of a combinatorial optimization. Then, we apply the characterization to popular network graph topologies (Erdős-Rényi, planted partition and power law graphs) as well as general ones, for asymptotically computing the diffusion time for those graphs. Our analysis shows that the diffusion time undergoes two levels of order-wise reduction, where the first and second one are solely contributed by the number of seeded users, i.e., quantity, and the amount of incentives, i.e., quality, respectively. In other words, it implies that the best strategy given budget is (a) first identify the minimum seed set depending on the underlying graph topology, and (b) then assign largest possible incentives to users in the set. We believe that our theoretical results provide useful implications and guidelines for designing successful advertising strategies in various practical applications.**

I. Introduction

With arise of online social networking services such as Facebook and Twitter, customers are more actively using their social network to exchange their opinions about new products. Simultaneously, companies can easily access corresponding data on people's reactions to the new products, which provides useful insights and opportunities on their marketing strategies. Motivated by this, the problem of selecting a subset of influential individuals, called *seeding problem*, has been extensively studied in the last decade [1]–[5], where the objective is to trigger the largest adoption of new products over social networks by seeding the influential subset, called *seed set*, i.e., providing some additional incentive for them to pre-adopt the new product.

To address the seeding problem, various diffusion models have been proposed, where we can broadly classify them into epidemic-based ones, e.g., [2], [6]–[12] and game-based ones, e.g., [3], [5], [9], [13], [14], [14]–[18] depending on how individuals interact with each other. In this paper, we adopt a game-based model where each individual strategically selects a product maximizing its utility depending on compatibility with others under social relationships with some noise. This model corresponds to a networked coordination game in which users update their states according to a noisy best-response. Seeding problem in the game-based diffusion model has been studied in non-progressive [5] or progressive setup [18].[1] But the previous work considers only a "strong" seeding in the sense that seeded users immediately adopt the new product, simply by starting at it, and they stay at the new product throughout their lifetime. However, in practice, seeded users often adjust the degree of their willingness to diffuse, depending on how much incentive is given to them. For example, one will differently behave between when she is paid \$100 and \$1000 by a company for advertising it. Hence, given a limit of seeding budget, the company needs to decide whether seeding more users with less incentive or less users with more incentive.

The major goal of this paper is to develop a new variant of the game-based diffusion model to reflect such a willingness of seeded users to diffuse the new product as a function of the given incentive, and then theoretically study how fast or slow the diffusion occurs, where the diffusion speed can be significantly different depending on *seeding strategy* consisting of not only the selection of seed set but also the amount of incentive to each seeded user. With the entire budget being constrained, the seed set selection and the incentive are coupled since the amount of incentive determines the size of seed set. Hence we need to strike a good balance between those two. The main contribution of this paper reveals on how we should use our given seeding budget between seed set and incentive, where more detailed summary of our contribution is provided in what follows:

- ***New model and characterization of diffusion time.*** We propose a new game-based model, called *incentivized game-diffusion model*, that includes the parameter α, corresponding to the amount of incentive provided for seeded users. We include this α as an additional payoff gain in the two-person coordination game whose aggregation over the neighbors is used as one's total payoff in the networked coordination game. This, seemingly a small variant of the traditional game-based model, incurs technical challenges

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[1]Progressive users refers the ones who are forced to stay at the new technology once they adopt it, whereas non-progressive users can switch between the new and the old technologies.

due to the need of jointly considering both progressive and non-progressive features by users as well as the diffusion incentive α, all in one framework. Despite these technical challenges, we combinatorially characterize the diffusion time which allows us to quantify the diffusion speed for our control knob (C, α), where C is the seed set and α is the incentive, as explained next.

- ◦ ***Diffusion time for random graphs.*** To build intuition, we study seeding strategy in three popular random graphs: Erdős-Rényi (ER), planted partition (PP), and power law (PL) graphs, which differ in terms of topological symmetry and the amount of degree bias. We provide the asymptotic analysis of the diffusion time as a function of C and α with C being chosen by topology-dependent seeding algorithms. Our asymptotic analysis provides the seed set size for each random graph that is sufficient to achieve an order-wise reduction of the diffusion time from that without seeding, i.e., an upper bound of the minimum seed set size for faster diffusion. In addition, we also obtain the necessary seed set size for the reduction. Interestingly, this sufficient and necessary seed set sizes are *independent* of the incentive α, where the latter affects only on the amount of reduction. Furthermore, we also prove that our bounds are tight under some conditions. Therefore, the diffusion time undergoes a sharp *phase transition* between the necessary and sufficient seed set sizes, and the amount of reduction of diffusion time is affected by the incentive α.
- ◦ ***Seeding algorithm for general graphs.*** We also obtain similar results for general graphs. Note that for ER, PP, and PL graphs, we perform our analysis using three different topology-dependent seeding algorithms. By unifying the intuitions we built in studying such random graphs, we propose a new topology-adaptive seeding algorithm applicable to arbitrary graphs, and also establish the corresponding upper bound on the minimum seed set size required for the order-wise reduction on the diffusion time. Obtaining such a result is more challenging compared to the case of random graphs. To do so, we use coupling techniques on several random processes and applying the meta-stability theory to obtain tractable upper and lower bounds on the diffusion time for general graph.

Our theoretical results for random and general graphs imply that the best strategy given budget is (a) first identify the minimum seed set, i.e., quantity, depending on the underlying graph topology, and (b) then assign largest possible incentives, i.e., quality, to users in the set. Such theoretical findings are also demonstrated in our extensive simulations with the dataset of real-world networks. We believe that our results provide useful implications and guidelines for designing successful advertising strategies in various practical applications.

A. Related Work

The diffusion models in literature can be broadly classified into: epidemic-based ones, e.g., [2], [6]–[12] and game-based ones, e.g., [9], [13]–[16] depending on how diffusion dynamically occurs. We primarily focus on summarizing the related work on game-based approaches due to its close similarity to our work.

We first summarize the work that uses a noisy best-response. The main focus in the early literature has been on the convergence of individuals' decisions to an equilibrium state. Using the notion of risk dominance (which roughly means more coordination gain for a new product), [13]–[15] focused on a simple condition on the structure of the payoff and the underlying graph topology for the new product to become widespread. Much later, the authors in [17], [18] characterized the convergence speed by a combinatorial optimization and established the asymptotic orders of diffusion time (without seeding) under various topologies. It has been also studied [5] how to accelerate the diffusion by seeding a subset of individuals, where their focus was developing "optimal" seeding algorithms for certain structured graphs. In [19], the authors provide some insights on the tradeoff between investing resource in improving quality of the product or marketing the product via studying spread of the product in a game-based diffusion model.

Other related work based on the game-based diffusion include those with the pure best-response dynamic. In this literature, people have mainly focused on finding a minimal seed set, often referred to as a *contagion set,* which makes a "widespread" cascade of rational adoptions of the new product [3], [9], [20], where this problem is also called *target set selection*. Morris [3] focused on providing topological conditions under which there exists a finite size of contagion set with growing network size. Kleinberg [9] studied the impact of progressiveness, i.e., once a user adopts the new product, she keeps using the adoption, by comparing the contagion sets in the progressive and non-progressive cases. Ackerman *et al.* [20] proposed a randomized algorithm that finds a contagion set, where an upper bound of the minimal contagion set is studied.

Our work differs from the above first in the model: In the papers [5], [18] with seeding under the noisy best-response dynamics, they assume that the seeding effect is *immediate* in the sense that a seeded node is convinced to adopt before the actual diffusion starts, while our model is more general in the sense that seeded users may still be strategic and rational as a function of the given incentive α. This difference in modeling enables us to address new, practical questions. In relation to the contagion-set research with the pure best-response dynamic [3], [20]–[22], a similar goal is taken: under what condition, efficient diffusion occurs, but due to the difference in modeling, they focus on the minimum seeding for the widespread, whereas our interest lies in the minimum (C, α) for an order-wise reduction of diffusion time.

II. Model and Preliminaries

We consider a social network as an undirected graph $G = (V, E)$ with $|V| = n$, where V and E are the sets of nodes and edges, respectively. Each node $i \in V$ represents an individual (or *player*) and its state is denoted by $x_i \in \{-1, +1\}$, where

$x_i = +1$ (resp. $x_i = -1$) means that node i adopts the new product (resp. the old one). Each edge represents a social relationship between two individuals whose states affect each other.

A. Networked Coordination Game

Each strategic user in the social network determines her state by playing a certain kind of game with its neighbors. To formally discuss, we describe the *payoffs* of individuals, where an individual's payoff is affected by its neighbors' strategies. We first consider the well-known two-person coordination game whose payoff matrix is given by Table I(a), where an individual can choose one of new or old product, i.e., $+1$ or -1. We make the following natural assumptions on the payoffs. First, there always exists coordination gain, i.e., $a > d$ and $b > c$. Second, coordination gain becomes larger for the new technology, i.e., $a - d > b - c$. For the convenience of notation, let us define $h := \frac{a-d-b+c}{a-d+b-c}$ and $h_i := h|N(i)|$ where $N(i)$ is the set of node i's neighbors, i.e., $N(i) = \{j \in V \mid (i,j) \in E \text{ and } i \neq j\}$. Without loss of generality we use a normalized payoff matrix with h in Table I(b).

TABLE I: Payoff matrix $P(x_i, x_j)$ of an unseeded node i

(a) Original

$x_i \backslash x_j$	$+1$	-1
$+1$	a	c
-1	d	b

(b) Normalized

$x_i \backslash x_j$	$+1$	-1
$+1$	$1+h$	0
-1	0	$1-h$

We now extend the two-person coordination game to its *networked* version as follows: let $\boldsymbol{x} = (x_i \in \{-1,+1\} : i \in V)$, and $\boldsymbol{x}_{-i} = (x_j : j \in V \setminus \{i\})$ denote state of the entire nodes and state of them except for node i, respectively. Then, in the n-person game over G, node i's payoff $P_i(x_i, \boldsymbol{x}_{-i})$ for state $\boldsymbol{x}$ is simply the aggregate payoff against all of i's neighbors, i.e., $P_i(x_i, \boldsymbol{x}_{-i}) = \sum_{j \in N(i)} P(x_i, x_j)$, where $P(x_i, x_j)$ is the payoff from the two-person coordination game in Table I.

B. Diffusion Dynamics

In this section, we describe the diffusion dynamic, i.e., how the new products are dynamically spread over the social network G over time. We assume that each individual has its own independent Poisson clock with unit rate, and whenever the clock ticks, it decides which product to adopt according to its diffusion dynamics. As will be elaborated shortly, each individual also updates its state, depending on whether it is seeded or not. Let $C \subset V$ denote the set of seeded nodes, often called *seed set*, and we assume that the diffusion starts after one selects the seed set C. Then, when $\boldsymbol{x}$ is the state at the moment of node i's update, node i determines its state with the following probabilities:

$$\mathbb{P}[s_i|\boldsymbol{x}] = \begin{cases} \text{logit}(h, s_i, x) & \text{if } i \notin C, & \text{(1a)} \\ \text{logit}(h+\alpha, s_i, \boldsymbol{x}) & \text{if } i \in C \text{ and } x_i = -1, & \text{(1b)} \\ \mathbb{1}_+(s_i) & \text{if } i \in C \text{ and } x_i = +1, & \text{(1c)} \end{cases}$$

where we let $\mathbb{1}_+(s)$ indicate $s = +1$, i.e., $\mathbb{1}_+(s) = 1$ if $s = +1$ and $\mathbb{1}_+(s) = 0$ otherwise, and we define

$$\text{logit}(h, s_i, \boldsymbol{x}) := \frac{\exp(\beta s_i K_i(h, \boldsymbol{x}))}{\exp(\beta K_i(h, \boldsymbol{x})) + \exp(-\beta K_i(h, \boldsymbol{x}))}, \quad (2)$$

and $K_i(h, \boldsymbol{x}) := h_i + \sum_{j \in N(i)} x_j$.

An unseeded node $i \notin C$ changes its state with the probability in (1a), which can be interpreted as a noisy-version of the best response dynamics as follows: In the networked coordination game, for a given state $\boldsymbol{x}$, the unseeded node i's best strategy to maximize its payoff is choosing

$$\text{sign}\big(P_i(+1, \boldsymbol{x}_{-i}) - P_i(-1, \boldsymbol{x}_{-i})\big) = \text{sign}\big(K_i(h, \boldsymbol{x})\big),$$

since $P_i(+1, \boldsymbol{x}_{-i}) - P_i(-1, \boldsymbol{x}_{-i})$ (i.e., $K_i(h, \boldsymbol{x})$) is the payoff difference between choosing $+1$ and -1. However, in practice, people are often affected by many external and internal noise factors in their decision making. To model such noise, we introduce a small mutation probability that a state is irrationally chosen, often called noisy best response. We consider *logit-response dynamics* [5], [18], [23]–[26] that individuals adopt a product according to a distribution of the logit form which allocates larger probability to product delivering larger payoffs. The parameter β in (2) represents the degree of rationality, where $\beta = \infty$ and $\beta = 0$ correspond to the best response and the random response, respectively. We focus on the case of that users are sufficiently rational, i.e., large β regime.

A seeded node $i \in C$ would be incentivized to adopt the new product with more probability, which is modeled by two factors: (i) aggressiveness in (1b) and (ii) progressiveness in (1c). First, we assume that before adopting the new product, the seeded node i is provided the additional payoff $\alpha > 0$ to choose $+1$ in the normalized payoff matrix[2] as incentive, i.e., the values of $P_i(x_i, x_j)$ at $(+1,+1)$ and $(+1,-1)$ become $1 + h + \alpha$ and α, respectively, then its noisy best response corresponds to the probability in (1b), where it selects $+1$ more aggressively due to the incentive $\alpha > 0$, i.e., $P_i(+1, \boldsymbol{x}_{-i}) - P_i(-1, \boldsymbol{x}_{-i}) = K_i(h + \alpha, \boldsymbol{x})$. Second, we assume that once the seeded node i accepts the new product with the incentive, it becomes progressive and it keeps choosing $+1$ irrespective of neighbors' decisions, where the corresponding dynamics is described in (1c).

Difference from prior work. We note that for $\beta > 0$, a seeded node does not necessarily mean that it immediately adopts the new product in our model if $\alpha < \infty$, while most prior works [2]–[5], [18] in the literature considered the case $\alpha = \infty$ which is a special case of our settings since $\lim_{\alpha \to \infty} \text{logit}(h + \alpha, +1, \boldsymbol{x}) = 1$. The assumption $\alpha < \infty$ is more realistic than $\alpha = \infty$ since even a seeded individual acts strategically in practice. In addition, the analysis of our model with finite α is technically more challenging than those in prior work because by fixing the seeded users at $+1$ and truncating the diffusion over the unseeded users only, it is enough to study a single type of users, either progressive [18] or non-progressive [5], whereas our model includes a mixture of progressive seeded users and non-progressive unseeded users.

[2]In the original payoff matrix, the additional payoff is $\frac{(a-d+b-c)\alpha}{2} > 0$.

C. Diffusion Time

Given $\mathcal{A} = (C, \alpha)$, the random process according to the diffusion dynamics can be viewed as a continuous Markov chain $\mathcal{M}_{\mathcal{A}}$ with the state space $\mathcal{X} := \{-1, +1\}^V$. All nodes in the seed set C will stay at $+1$ once it adopts $+1$ and other unadvertised nodes are allowed to oscillate between -1 and $+1$ according to the logit dynamics. Hence $\mathcal{M}_{\mathcal{A}}$ is not time-reversible but the truncation of $\mathcal{M}_{\mathcal{A}}$ on $\mathcal{X}_C := \{\boldsymbol{x} \in \mathcal{X} \mid x_i = +1 \ \forall i \in C\}$ is time-reversible with the stationary distribution $\mu_{\mathcal{A}}(\boldsymbol{x})$:

$$\mu_{\mathcal{A}}(\boldsymbol{x}) = \begin{cases} \frac{1}{Z_C} \exp(-\beta H_{\mathcal{A}}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \in \mathcal{X}_C \\ 0 & \text{otherwise} \end{cases}$$

where $Z_C := \sum_{\boldsymbol{x} \in \mathcal{X}_C} \exp(-\beta H(\boldsymbol{x}))$ and with $\alpha_i := \alpha |N(i)|$,

$$H_{\mathcal{A}}(\boldsymbol{x}) := -\sum_{(i,j) \in E} x_i x_j - \sum_{i \in V} h_i x_i - \sum_{i \in C} \alpha_i x_i. \quad (3)$$

We note that $H_{\mathcal{A}}(\boldsymbol{x})$, often called *potential function*, has a unique minimizer at the state of all $+1$, denote by $\mathbf{+1}$, regardless of C and α, if $h > 0$. This implies that the entire network would adopt the new product in the long run, where the diffusion speed depends on the choice of $\mathcal{A} = (C, \alpha)$.

Definition of diffusion time. To measure the speed of diffusion, we use the hitting time to $\mathbf{+1}$. Formally, we define a couple of related concepts. First, a random variable called the *hitting time* of our random process under $\mathcal{A}$ from an initial state $\boldsymbol{z} \in \mathcal{X}$, and denote it by $T_{\mathcal{A}}(\boldsymbol{z})$:

$$T_{\mathcal{A}}(\boldsymbol{z}) := \inf\{t \geq 0 \mid \boldsymbol{x}(t) = \mathbf{+1}, \boldsymbol{x}(0) = \boldsymbol{z}\}.$$

We next define the typical value of the hitting time, called *diffusion time* $\tau_{\mathcal{A}}(G)$:

$$\tau_{\mathcal{A}}(G) := \sup_{\boldsymbol{z} \in \mathcal{X}} \inf \{t \geq 0 \mid \mathbb{P}[T_{\mathcal{A}}(\boldsymbol{z}) \geq t] \leq 1/e\}. \quad (4)$$

This means that with probability more than $1 - 1/e > 1/2$, every node adopts the new product $+1$ within time $\tau_{\mathcal{A}}(G)$ from any initial state.

III. Main Result

We start by providing the characterization of diffusion time in (4), which enables us to study our main question of under what conditions of seed set C and incentive α, i.e., $\mathcal{A} = (\mathcal{C}, \alpha)$ the diffusion becomes fast or slow for large β.

A. Characterization of Diffusion Time

Theorem 1: As $\beta \to \infty$, for given $\mathcal{A} = (C, \alpha)$, diffusion time $\tau_{\mathcal{A}}(G)$ is

$$\tau_{\mathcal{A}}(G) = \exp(2\beta \cdot \Gamma_{\mathcal{A}}(G) + o(\beta)).$$

In the above, $\Gamma_{\mathcal{A}}(G)$ is defined as follows:

$$\Gamma_{\mathcal{A}}(G) := \max_{Z \subset V} \min_{\underline{v} \in \mathcal{L}(V \setminus Z)} \max_{t \leq T_C(\underline{v})} [H_{\mathcal{A}}(V_t \cup Z) - H_{\mathcal{A}}(Z)] \quad (5)$$

where for a subset $S \subset V$, we define $\mathcal{L}(S)$ as the set of all vertex orderings of S, and $H_{\mathcal{A}}(S)$ is

$$H_{\mathcal{A}}(S) := \text{cut}(S, V \setminus S) - \sum_{i \in S} h_i - \sum_{i \in S \cap C} \alpha_i, \quad (6)$$

and for an ordering $\underline{v} = (v_1, ..., v_{|\underline{v}|})$, we let $V_t := \{v_1, ..., v_t\}$ and $T_C(\underline{v}) := \min\{1 \leq t \leq |\underline{v}| \mid v_t \in C\} \cup \{|\underline{v}|\}$.

The proof is given in Section IV-A. This type of characterization has been made in other related work [5], [17], [18], where they refer to $\Gamma_{\mathcal{A}}(G)$ as *diffusion exponent*, which depends on seed set C, incentive α and graph G. For large β, it suffices to study this diffusion exponent which is our focus of this paper. We comment that our characterization of the diffusion time generalizes the ones in [5], [17] and [18] each of which is a special case of ours for $\mathcal{A} = (\emptyset, \infty)$, $\mathcal{A} = (C, \infty)$ and $\mathcal{A} = (V, 0)$, respectively.

To help the readers with understanding the intuition of $\Gamma_{\mathcal{A}}(G)$, regard the sequence of subsets $\underline{S} = \{S_0 = Z, ..., S_T = V\}$ as the path $\underline{\boldsymbol{\omega}} = \{\boldsymbol{\omega}_0 = \boldsymbol{z}, ..., \boldsymbol{\omega}_T = \mathbf{+1}\}$ where S_t is the set of nodes adopting $+1$ at $\boldsymbol{\omega}_t$ so that $H_{\mathcal{A}}(\boldsymbol{\omega}_t) = 2H_{\mathcal{A}}(S_t) + \text{some constant}$. The main intuition is as follows: the dynamics of the Markov chain $\mathcal{M}_{\mathcal{A}}$ has a tendency to decrease the value of the potential function $H_{\mathcal{A}}$, but to reach the global minimizer $\mathbf{+1}$ from the initial state $\boldsymbol{z}$, it may be necessary to go through the states with high values of $H_{\mathcal{A}}$. These states create a barrier and the hitting time is an exponential function of the height along the most probable path which has the smallest barrier among all paths from $\boldsymbol{z}$ to $\mathbf{+1}$. The similar interpretation is also given in [17] under a purely non-progressive setting without seeding, but in our diffusion model, the aggressiveness of seeded users in (1b) is captured by the last term of the potential function $H_{\mathcal{A}}$ in (5) and the progressiveness of them in (1c) is captured by $T_C(\underline{v})$ in the last max of (5).

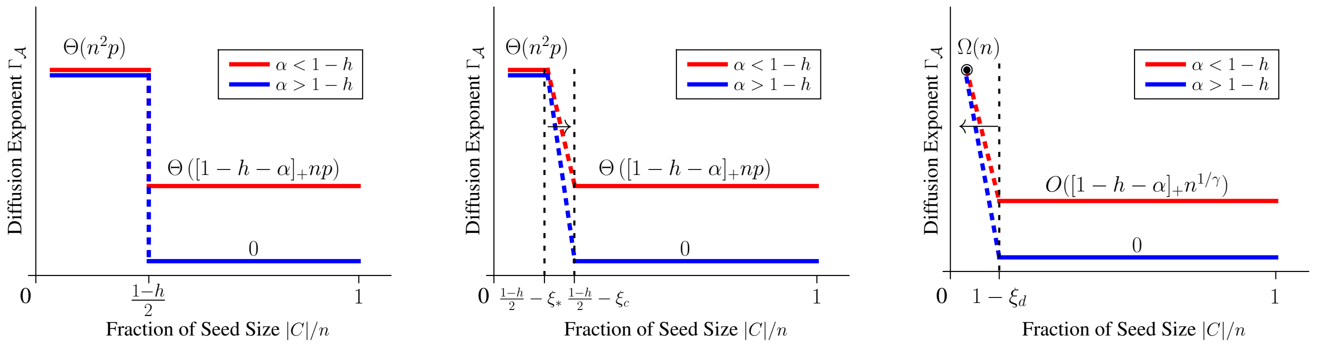
Our goal is to understand how a seeding strategy $\mathcal{A} = (C, \alpha)$ affects the speed of diffusion, with particular focus on the role of each of C and α. In Section III-B, we consider popular random graphs for each of which we apply a topology-dependent seed set selection strategy, partially motivated by the results in [5], and in Section III-C, we consider a general graph, thus without any topological information a priori, where we apply a seed set selection strategy that implicitly learns and exploits its graphical structure.

B. Analysis of Diffusion Exponent: Popular Random Graphs

We describe three popular random graphs which we consider in this paper, and present the seed set selection strategy applied in each graph, when the seed set size is given by k.

(a) *Erdős-Rényi (ER) graph* with parameter (n, p) is a random graph consisting of n nodes where each node pair has an edge with probability p. As a seed set selection strategy, we consider a strategy that selects k nodes uniformly at random, named $\textsf{Random}(k)$.

(b) *Planted partition (PP) graph* with parameter $(n, p, q, \boldsymbol{\alpha})$ is a random graph where total n nodes are divided into m disjoint clusters $\{V_1, ..., V_m\}$ and each cluster V_l consists of ω_l-fraction of nodes, i.e., $|V_l| = \omega_l n$ and $\sum_{l=1}^{m} \omega_l = 1$, where $\omega_1 \geq \omega_2 ... \geq \omega_m$. Every node pair $i, j \in V$ has an edge with probability p if the nodes are in the same cluster, i.e., $i, j \in V_l$, and with probability q otherwise. As



(a) Erdős-Rényi graph under arbitrary seeding.

(b) Planted partition graph under the cluster-based seeding ($|\xi_* - \xi_c| \to 0$ as $\frac{q}{p} \to 0$).

(c) Power law graph under the degree-based seeding ($\xi_d \to 1$ as $\gamma \to \infty$).

Fig. 1: A graphical summary of Theorems a, b, and c, where constants ξ_*, ξ_c, and $\xi_d > 0$ are defined therein.

a seed set selection strategy, we consider a cluster-based one, named $\mathsf{Cluster}(k)$ that chooses a set C of size k that is the solution of the following optimization problem:

$$\min_{C':|C'|=k} \max_{1\le l\le m} \left[\frac{1-h}{2}\omega_l - \delta_l \frac{q}{p} - \frac{|C' \cap V_l|}{n}\right] \quad (7)$$

where $\delta_l := \sum_{l'=1}^{l-1} \omega_{l'} - \frac{1-h}{2}(1-\omega_l)$.

(c) *Power law (PL) graph* with parameter (n, γ) has a power law degree distribution parameterized by $\gamma \ge 2$, i.e., the fraction of nodes having degree d is proportional to $d^{-\gamma}$. We consider the PL graph generated by the preferential attachment with minimum degree 2 as in [27], [28]. As a seed set selection strategy, we consider a degree-based one, named $\mathsf{Degree}(k)$, which chooses k nodes in decreasing order of their degrees.

We now present Theorem 2 which provides the asymptotic quantification of the diffusion exponents for three graphs. For each graph, we have two parts: part (i) provides a lower bound of the *necessary size* of seed set for the order-wise reduction of the diffusion exponent, and part (ii) provides an upper bound of the *sufficient size* of seed set for the same reduction. The proof is given in Section IV-B.

Theorem 2: As $n \to \infty$, for given $\alpha \ge 0$ and any small constant $\varepsilon > 0$, the following events occur almost surely:

(a) Suppose G is an ER graph with $np = \omega(\log n)$.
(i) For every seed set C such that $\frac{|C|}{n} \le \frac{1-h}{2} - \varepsilon$,

$$\Gamma_{\mathcal{A}}(G) = \Theta(n^2 p).$$

(ii) If seed set C is selected by $\mathsf{Random}(k)$ and $\frac{|C|}{n} \ge \frac{1-h}{2} + \varepsilon$, then

$$\Gamma_{\mathcal{A}}(G) = \begin{cases} \Theta((1-h-\alpha)np) & \text{if } \alpha \le 1-h-\varepsilon \\ 0 & \text{if } \alpha \ge 1-h+\varepsilon. \end{cases}$$

(b) Suppose G is a PP graph with $(1-\alpha_l)nq \le \alpha_l np = \omega(\log n)$ and $\alpha_m n = \Omega(n)$.
(i) For every C such that $\frac{|C|}{n} \le \frac{1-h}{2} - \xi_* - \varepsilon$,

$$\Gamma_{\mathcal{A}}(G) = \Theta(n^2 p)$$

where $\xi_* := m\sqrt{\frac{q}{p}}$.
(ii) If seed set C is selected by $\mathsf{Cluster}(k)$ and $\frac{|C|}{n} \ge \frac{1-h}{2} - \xi_c + \varepsilon$, then

$$\Gamma_{\mathcal{A}}(G) = \begin{cases} \Theta((1-h-\alpha)np) & \text{if } \alpha \le 1-h-\varepsilon \\ 0 & \text{if } \alpha \ge 1-h+\varepsilon \end{cases}$$

where $\xi_c := \frac{(1-h)^2}{2} \frac{q}{p}$.

(c) Suppose G is a PL graph with $\gamma > 1$ and $d_{\min} = 2$.
(i) If $C = \emptyset$, i.e., $|C| = 0$, then for small enough $h \ge 0$,

$$\Gamma_{\mathcal{A}}(G) = \Omega(n)^3.$$

(ii) If seed set C is selected by $\mathsf{Degree}(k)$ and $\frac{|C|}{n} \ge 1 - \xi_d + \varepsilon$, then

$$\Gamma_{\mathcal{A}}(G) = \begin{cases} O\left((1-h-\alpha)n^{\frac{1}{\gamma}}\right) & \text{if } \alpha \le 1-h-\varepsilon \\ 0 & \text{if } \alpha \ge 1-h+\varepsilon \end{cases}$$

where $\xi_d := \frac{1}{2^\gamma(\zeta(\gamma)-1)}$ with $\zeta(\gamma) := \sum_{d=1}^{\infty} \frac{1}{d^\gamma}$.

In Figure 1, we graphically summarize the above theorem which shows a *phase transition* of the diffusion exponent in an interval of the seed set size for each graph. For example, in ER graphs, any seed set C cannot reduce the order of the diffusion exponent if $\frac{|C|}{n} \le \frac{1-h}{2} - \varepsilon$. But we can reduce the order with seed set C such that $\frac{|C|}{n} \ge \frac{1-h}{2} + \varepsilon$. Hence, in ER graphs, the phase transition from $\Theta(n^2p)$ to $\Theta((1-h-\alpha)np)$ occurs in $\left[\left(\frac{1-h}{2} - \varepsilon\right)n, \left(\frac{1-h}{2} + \varepsilon\right)n\right]$, of which the minimum value is a lower bound on *necessary size* and the maximum one is an upper bound on *sufficient size* for the phase transition.

Seed set size first and then incentive. We first focus on the interpretation of Theorem 2(a) for ER graphs, where analogous interpretation also works for PP and PL graphs. In ER graphs, it is not possible to reduce the order of the diffusion exponent by any seed set even with $\alpha = \infty$, if the size of seed set does not exceed a certain threshold, i.e., $|C| < \frac{1-h}{2}n$. In other words, it is necessary to have a seed set of size more than $\frac{1-h}{2}n$ for the order-wise reduction while having large α is not

[3]This diffusion exponent without seeding in PL graphs was studied in [17].

necessary, i.e., we need to secure a certain number of seeds first instead of high incentive. However, once the size of seed set even slightly exceeds the threshold, i.e., $|C| > \frac{1-h}{2}n$, there is a phase transition of the diffusion exponent from $\Theta(n^2 p)$ to $\Theta((1-h-\alpha)np)$ where only incentive α can reduce the order of the diffusion exponent. This implies that after the phase transition, it is more efficient to increase the incentive than to increase the seed set size. We note that $\alpha = 1-h$ makes the diffusion extremely fast, i.e., $\Gamma_{\mathcal{A}}(G) = 0$, since $1-h$ is nothing but the minimum of additional payoff that makes a seeded user's best response to be adopting the new product regardless of its neighbors' choice.

Selection of seed set. As we explained in the above, Theorem 2(a) shows a sharp phase transition of the diffusion exponent at the threshold of the seed size in ER graphs, where due to the symmetric connectivity, any arbitrary choice of seed set C can have such phase transition at $\frac{|C|}{n} = \frac{1-h}{2}$. Such narrow gap between the necessary and the sufficient sizes implies the efficiency of $\mathsf{Random}(k)$ for ER graphs, which is also shown in [5]. In Theorems 2(b) and 2(c), we also obtain similar phase transitions between $\frac{1-h}{2} - \xi_* < \frac{|C|}{n} < \frac{1-h}{2} - \xi_c$ for PP graphs and $0 < \frac{|C|}{n} < 1 - \xi_d$ for PL graphs. The phase transition in PP graphs and PL graphs become sharp as the one in ER graphs when the fraction of the inter-cluster edges decreases, i.e., $\frac{q}{p} \to 0$ for PP graphs and the degree distribution becomes more skewed, i.e., $\gamma \to \infty$. Different from ER graphs, for PP and PL graphs, in order to have a phase transition, we need to carefully select seeds depending on the network topology: $\mathsf{Cluster}(k)$ for PP graphs and $\mathsf{Degree}(k)$ for PL graphs.

We note that $\mathsf{Cluster}(k)$ is analogous to that in [5], but we improve the previous one for a tighter phase transition, in terms of that using the previous one, $|C| = \frac{1-h}{2}n$ is necessary to reduce the order of the diffusion exponent, i.e., $\mathsf{Cluster}(k)$ saves budget $\xi_c n$ for the same purpose, where the improvement is made by the second term in (7). The intuition behind $\mathsf{Cluster}(k)$ with (7) is collecting more seeds from larger clusters but with some balance between the fraction of seeds in each cluster since a larger cluster has higher $\frac{1-h}{2}\omega_l - \delta_l \frac{q}{p}$ l. The PL graph is generated by the preferential attachment mechanism, i.e., the more connected node has more likely to receive new links. Thus connectivity is concentrated in a small number of high-degree nodes like hubs connecting other low-degree nodes. Thus it is very natural to seed the high degree nodes as $\mathsf{Degree}(k)$ does.

C. Analysis of Diffusion Exponent: General Graphs

In the previous section, we studied popular random graphs and the corresponding seeding algorithms, i.e., the arbitrary (or random) seeding for ER graphs, the cluster-based seeding for PP graphs, and the degree-based seeding for PL graphs. To tolerate topology-sensitive performances of seeding algorithms, we propose a new one, called $\mathsf{General}(k)$, working for arbitrary graphs, which is inspired by those in the previous sections.

Algorithm 1: $\mathsf{General}(k)$

Input: Graph $G = (V, E)$ and seed budget k

Output: Seed set C

1 Start with $V_0 \leftarrow V$.
2 **for** $t = 1, 2, ..., n-k$ **do**
3 Set $W_{t-1} \leftarrow \{i \in V_{t-1} \mid \frac{1-h}{2}|N(i)| \le |N(i) \cap V_{t-1}|\}$
4 **if** $W_{t-1} \neq \emptyset$ **then**
5 Pick node i_t from W_{t-1} such that $i_t \in \arg\max_{i \in W_{t-1}} \frac{\frac{1-h}{2}|N(i)|}{|N(i) \cap V_{t-1}|(|N(i) \cap V_{t-1}|+1)}$
6 **else**
7 Pick node i_t from V_{t-1} such that $i_t \in \arg\max_{i \in V_{t-1}} \frac{\frac{1-h}{2}|N(i)|}{|N(i) \cap V_{t-1}|(|N(i) \cap V_{t-1}|+1)}$
8 **end**
9 Update $V_t \leftarrow V_{t-1} \setminus \{i_t\}$.
10 **end**
11 Output $C \leftarrow V_{n-k}$

The formal description of the proposed algorithm is in Algorithm 1, where its running time is $O(n^2)$. It iteratively narrows down from V of size n to output C of size k, by greedily removing node i_t that has (nearly) minimal influence to V_t, i.e., $|N(i_t) \cap V_{t-1}|$ at each iteration (see line 5). One can observe that for seed size $k \ge (\frac{1-h}{2}n - \xi_c)n$, the algorithm is identical to the cluster-based algorithm for PP graphs and for seed size $k \ge (1-\xi_d)n$, its output C is identical to the degree-based algorithm for PL graphs. Namely, one can restate Theorem 2 under $\mathsf{General}(k)$ while maintaining the same upper and lower bounds on the necessary size and the sufficient size. Furthermore, for general graphs, we establish an upper bound on the sufficient size to reduce the diffusion exponent at certain order in the following theorem whose proof is provided in the extended version of this paper [29].

Theorem 3: For graph $G = (V, E)$, define

$$\kappa := \sum_{i \in V} \frac{\lceil \frac{1-h}{2}|N(i)| \rceil}{|N(i)|+1}. \tag{8}$$

If seed set C is selected by $\mathsf{General}(k)$ and $|C| \ge \kappa$, then for given $\alpha \ge 0$ and any small constant $\varepsilon > 0$,

$$\Gamma_{\mathcal{A}}(G; C) = \begin{cases} O((1-h-\alpha)d_{\max}) & \text{if } \alpha \le 1-h-\varepsilon \\ 0 & \text{if } \alpha \ge 1-h+\varepsilon \end{cases}$$

where $d_{\max} = \max_{i \in V} |N(i)|$.

It is not hard to check that κ matches the upper bound on the sufficient size for ER graphs in Theorem 2(a) and it is slightly larger than the one for PP graphs in Theorem 2(a). We note that a node i's best response is the new product if the fraction of neighbors adopting the new product is larger than $\frac{\lceil \frac{1-h}{2}|N(i)| \rceil}{|N(i)|+1}$. Then the value of κ in (8) is the summation of the lower bound on the fraction for every node.

IV. Proof of Theorem

A. Proof of Theorem 1

We first construct a time-reversible Markov chain $\mathcal{M}'_{\mathcal{A}}$ based on the non-reversible chain $\mathcal{M}_{\mathcal{A}}$. In $\mathcal{M}'_{\mathcal{A}}$, an unseeded node $i \notin C$ has the same probability of the one in (1a) but a seeded node $i \in C$ has a positive probability to go back to -1 from $+1$, to guarantee the time-reversibility, as described in the following:

$$\mathbb{P}[s_i|\boldsymbol{x}] = \begin{cases} \text{logit}(h+\alpha, s_i, \boldsymbol{x}) & \text{if } x_i = -1 \\ \exp(-\beta\beta')\text{logit}(h+\alpha, s_i, \boldsymbol{x}) & \text{if } x_i = +1 \\ \quad + (1-\exp(-\beta\beta'))\mathbb{1}_+(s_i). \end{cases}$$

As $\beta \to \infty$, the transition probability in $\mathcal{M}'_{\mathcal{A}}$ with $\beta' > 0$ converges to that in $\mathcal{M}_{\mathcal{A}}$ thus the diffusion time $\tau'_{\mathcal{A}}$ of $\mathcal{M}'_{\mathcal{A}}$ would also converge to $\tau_{\mathcal{A}}$. We formally obtain the convergence of $\tau'_{\mathcal{A}}$ to $\tau_{\mathcal{A}}$ with sufficiently large β' in the following lemma whose proof is provided in the extended version of this paper [29].

Lemma 1: Suppose $\beta' \geq 8n^2$. Then, as $\beta \to \infty$, $\tau'_{\mathcal{A}}$ converges to $\tau_{\mathcal{A}}$.

Thus, we now focus on the characterization of $\tau'_{\mathcal{A}}(G)$. To do so, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$ such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are same except a node i, i.e., $x_j = y_j$ for all $j \neq i$ and $x_i = -y_i$, we write the stationary distribution $\mu'_{\mathcal{A}}(\boldsymbol{x})$ and the transition rate $p'_{\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y})$ of $\mathcal{M}'_{\mathcal{A}}$ as $\mu'_{\mathcal{A}}(\boldsymbol{x}) = \exp(-\beta H'_{\mathcal{A}}(\boldsymbol{x}) + o(\beta))$ and $p'_{\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) = \exp(-\beta V'_{\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) + o(\beta))$ where

$$H'_{\mathcal{A}}(\boldsymbol{x}) := H_{\mathcal{A}}(\boldsymbol{x}) - \beta' \sum_{i \in C} \mathbb{1}_+(x_i),$$

$$V'_{\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) := \begin{cases} [H_{\mathcal{A}}(\boldsymbol{y}) - H_{\mathcal{A}}(\boldsymbol{x})]_+ & \text{if } i \notin C \\ [H_{\mathcal{A}}(\boldsymbol{y}) - H_{\mathcal{A}}(\boldsymbol{x})]_+ + \beta' \mathbb{1}_+(x_i) & \text{if } i \in C, \end{cases}$$

and for $x \in \mathbb{R}$, $[x]_+ := \max\{x, 0\}$. These expressions of $\mu'_{\mathcal{A}}$ and $p'_{\mathcal{A}}$ allow us to apply the hitting time analysis of the Freidlin-Wentzell chain, provided in Chapter 6 [30], for the time-reversible chain $\mathcal{M}'_{\mathcal{A}}$ and we obtain the following lemma as a corollary of Theorem 6.38 therein.

Lemma 2: As $\beta \to \infty$, $\tau'_{\mathcal{A}} = \exp(\beta \Gamma'_{\mathcal{A}} + o(\beta))$ where $\Gamma'_{\mathcal{A}}(G)$ is

$$\min_{\underline{\boldsymbol{\omega}}:-\mathbf{1}\to+\mathbf{1}} \max_{t'<|\underline{\boldsymbol{\omega}}|} \max_{t' \leq t < |\underline{\boldsymbol{\omega}}|} [H'_{\mathcal{A}}(\boldsymbol{\omega}_t) + V'_{\mathcal{A}}(\boldsymbol{\omega}_t, \boldsymbol{\omega}_{t+1}) - H'_{\mathcal{A}}(\boldsymbol{\omega}_{t'})]. \tag{9}$$

Here the $\min$ runs over every possible path $\underline{\boldsymbol{\omega}}$ from $-\mathbf{1}$ to $+\mathbf{1}$ such that for $0 \leq t < |\underline{\boldsymbol{\omega}}|$, $\boldsymbol{\omega}_t$ and $\boldsymbol{\omega}_{t+1}$ are same except a node's state.

Thus it is enough to show that for sufficiently large $\beta' \geq 8n^2 \geq 4\max_{\boldsymbol{x}\in\mathcal{S}} H_{\mathcal{A}}(\boldsymbol{x})$,

$$\Gamma'_{\mathcal{A}}(G) = 2\Gamma_{\mathcal{A}}(G). \tag{10}$$

Recalling that for $\boldsymbol{z} \in \mathcal{S}$ and $Z \subset V$ such that $Z = \{i \in V \mid z_i = +1\}$, $H_{\mathcal{A}}(\boldsymbol{\omega}_t) = 2H_{\mathcal{A}}(S_t) + \text{some constant}$, we can rewrite $\Gamma_{\mathcal{A}}(G)$ in (5) as follows:

$$2\Gamma_{\mathcal{A}}(G) = \max_{\boldsymbol{z}\neq+\mathbf{1}} \min_{\underline{\boldsymbol{\omega}}:[\boldsymbol{z}\to+\mathbf{1}]_V} \max_{t \leq T_C(\underline{\boldsymbol{\omega}})} [H_{\mathcal{A}}(\boldsymbol{\omega}_t) - H_{\mathcal{A}}(\boldsymbol{z})] \tag{11}$$

where for a subset $S \subset V$, we let $[\boldsymbol{z} \to +\mathbf{1}]_S$ denote the set of every possible path $\underline{\boldsymbol{\omega}}$ from $\boldsymbol{z}$ to $+\mathbf{1}$ such that for all $i \in S$ and $0 \leq t' \leq t < |\underline{\boldsymbol{\omega}}|$, $\omega_{t,i} = +1$ if $\omega_{t',i} = +1$ and we let $T_C(\underline{\boldsymbol{\omega}}) := \min\{1 \leq t < |\underline{\boldsymbol{\omega}}| \mid \exists i \in C \text{ s.t. } \omega_{t-1,i} = -1 \text{ and } \omega_{t,i} = +1\} \cup \{|\underline{\boldsymbol{\omega}}|\}$.

To show (10), we will rewrite the optimization in (9) as the one in (11). Suppose $\underline{\boldsymbol{\omega}} \notin [-\mathbf{1} \to +\mathbf{1}]_C$. Then there exists s such that $\omega_{s,i} = +1$ and $\omega_{s+1,i} = -1$. Thus it follows that

$$\begin{aligned} &\max_{t'<|\underline{\boldsymbol{\omega}}|} \max_{t' \leq t < |\underline{\boldsymbol{\omega}}|} [H'_{\mathcal{A}}(\boldsymbol{\omega}_t) + V'_{\mathcal{A}}(\boldsymbol{\omega}_t, \boldsymbol{\omega}_{t+1}) - H'_{\mathcal{A}}(\boldsymbol{\omega}_{t'})] \\ &\geq \; V'_{\mathcal{A}}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s+1}) \; \geq \; \beta', \end{aligned}$$

which implies that we can reduce the search space of the $\min$ in (9) to $[-\mathbf{1} \to +\mathbf{1}]_C$ when β' is sufficiently large.

Suppose $\underline{\boldsymbol{\omega}} \in [-\mathbf{1} \to +\mathbf{1}]_C$ and there exists $T < |\underline{\boldsymbol{\omega}}|$ such that for some $i \in C$, $\omega_{T-1,i} = -1$ and $\omega_{T,i} = +1$. Then for all t, t' such that $t' \leq T < t$, it follows that

$$\begin{aligned} &H'_{\mathcal{A}}(\boldsymbol{\omega}_t) + V'_{\mathcal{A}}(\boldsymbol{\omega}_t, \boldsymbol{\omega}_{t+1}) - H'_{\mathcal{A}}(\boldsymbol{\omega}_{t'}) \\ &= H_{\mathcal{A}}(\boldsymbol{\omega}_t) + [H_{\mathcal{A}}(\boldsymbol{\omega}_{t+1}) - H_{\mathcal{A}}(\boldsymbol{\omega}_t)]_+ - H_{\mathcal{A}}(\boldsymbol{\omega}_{t'}) \\ &\quad - \beta' \sum_{i \in C} (\mathbb{1}_+(\omega_{t,i}) - \mathbb{1}_+(\omega_{t',i})) \end{aligned}$$

which is less than $-\frac{1}{2}\beta'$ since $\mathbb{1}_+(\omega_{t,i}) - \mathbb{1}_+(\omega_{t',i}) \geq 1$ and $\beta' \geq 4\max_{\boldsymbol{x}\in\mathcal{S}} H_{\mathcal{A}}(\boldsymbol{x})$. Thus, for sufficiently large β', we can reduce the search space of the last $\max$ in (9) and we obtain

$$\Gamma'_{\mathcal{A}}(G) = \min_{\underline{\boldsymbol{\omega}}:[-\mathbf{1}\to+\mathbf{1}]_C} \max_{t'<|\underline{\boldsymbol{\omega}}|} \max_{t' \leq t < T_C(\underline{\boldsymbol{\omega}})} [H_{\mathcal{A}}(\boldsymbol{\omega}_t) - H_{\mathcal{A}}(\boldsymbol{\omega}_{t'})]$$

where $T_C(\underline{\boldsymbol{\omega}}) := \min\{1 \leq t < |\underline{\boldsymbol{\omega}}| \mid \exists i \in C \text{ s.t. } \omega_{t-1,i} = -1 \text{ and } \omega_{t,i} = +1\} \cup \{|\underline{\boldsymbol{\omega}}|\}$. Furthermore, we can reduce the search space of the $\min$ from $[-\mathbf{1} \to +\mathbf{1}]_C$ to $[-\mathbf{1} \to +\mathbf{1}]_V$ by similar argument with the submodularity of $H_{\mathcal{A}}(S)$ used for the proof of Theorem 2 in [17]. This completes the proof.

B. *Proof of Theorem 2*

To prove Theorems 2(a)-(c) for different graphs and seed selections, we will use the following upper and lower bounds on $\Gamma_{\mathcal{A}}(G)$, whose proof is presented in the extended version of this paper [29].

Theorem 4 (Exponent Bound): For given G and seed set C, we define $\Gamma(C)$ as follows:

$$\Gamma(C) := \max_{C \subset Z \subset V} \min_{\underline{v} \in \mathcal{L}(V \setminus Z)} \max_{1 \leq t \leq |\underline{v}|} [H(V_t \cup Z) - H(Z)]\,. \tag{12}$$

where for a subset $S \subset V$, $H(S)$ is the value of $H_{\mathcal{A}}(S)$ with $\alpha = 0$. Then, for $\mathcal{A} = (C, \alpha)$, if follows that

$$\max\left\{\Gamma(C),\ \min_{i \in V}(1-h-\alpha)|N(i)|\right\} \tag{13}$$

$$\leq\ \Gamma_{\mathcal{A}}(G) \leq \max\left\{\Gamma(C),\ \max_{i \in C}(1-h-\alpha)|N(i)|\right\} \tag{14}$$

However, handling $\Gamma(C)$ directly in the above theorem is hard in general. So, we establish the following key lemmas that provide criteria to check if $\Gamma(C)$ is large or small, i.e., $\Gamma(C) \geq \delta$ or $\Gamma(C) = 0$, where the proofs are provided in the extended version of this paper [29].

Lemma 3: Consider graph $G = (V, E)$ and seed set C. Suppose that for a given k such that $|V \setminus C| \leq k \leq |V|$, there exists a constant $\delta > 0$ such that for every subset $S \subset V \setminus C$ with $|S| = k$,

$$\begin{aligned} &(1-h)\cdot\text{cut}(S, V\setminus S) - 2\cdot\text{cut}(S, C) - 2h\cdot\text{edge}(S) \\ &\quad = \ H(S \cup C) - H(C) \ \geq \ \delta, \end{aligned} \tag{15}$$

where $\text{edge}(S)$ is the number of edges between nodes in S, i.e., $\text{edge}(S) = |\{(i,j) \in E \mid i,j \in S\}|$. Then we have $\Gamma(C) \geq 2\delta$.

Lemma 4: Consider graph $G = (V, E)$ and seed set C. Suppose there exists a sequence $\underline{s}$ of nodes in $V \setminus C$ such that for all $t = 1, ..., |V \setminus C|$,

$$(1-h)|N(s_t)| - 2|N(s_t) \cap S_{t-1}| = H(S_t) - H(S_{t-1}) \leq 0 \quad (16)$$

where $S_t = C \cup \{s_1, ..., s_t\}$. Then we have $\Gamma(C) = 0$.

1) Proof of Theorem 2(a)*:* Recalling ER graph $G = (V, E)$ in Theorem 2(a), we note that $\varepsilon^2 np = \omega(\log n)$. Using the Chernoff inequality and the union bound, it follows that

$$\mathbb{P}\left[\bigcap_{i \in V}\left[\left||N(i)| - np\right| \leq \frac{\varepsilon}{2} np\right]\right] \geq 1 - O(n \exp(-\varepsilon^2 np)), \quad (17)$$

where the last term converges to 1 as $n \to \infty$ due to $\varepsilon^2 np = \omega(\log n)$. Then for large n, it follows that $|N(i)| = \Theta(np)$ and $\Gamma(G; C) = O(n^2 p)$ due to (12) with $H(S) \leq (1+2h)|E| = \Theta(n^2 p)$. Then using the above observations and Theorem 4, the proof is completed if the following events occur with high probability:

$$\Gamma(C) = \begin{cases} \Omega(n^2 p) & \text{if } \frac{|C|}{n} \leq \frac{1-h}{2} - \varepsilon, \quad (18) \\ 0 & \text{if } \frac{|C|}{n} \geq \frac{1-h}{2} + \varepsilon. \quad (19) \end{cases}$$

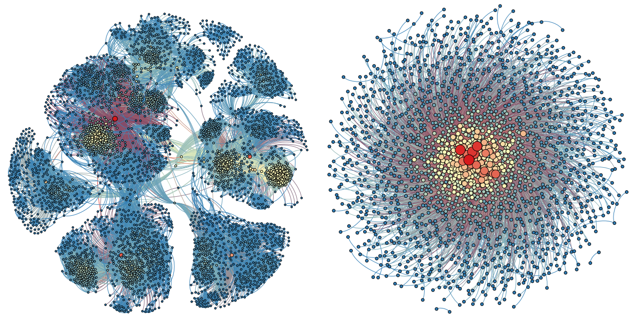
Due to space limitation, we provide the proofs of (18) and (19) without details which are provided in the extended version of this paper [29]. Suppose $\frac{|C|}{n} \leq \frac{1-h}{2} - \varepsilon$. Then, using the Chernoff inequality and the union bound, it is not hard to check that for every $S \subset V \setminus C$ such that $\frac{|S|}{n} = \frac{\varepsilon}{2}$, the values of $\text{cut}(S, V \setminus S)$, $\text{cut}(S, C)$, and $\text{edge}(S)$ concentrates at $\frac{\varepsilon}{2}\left(1 - \frac{\varepsilon}{2}\right) n^2 p$, $\frac{\varepsilon}{2}\left(\frac{1-h}{2} - \varepsilon\right) n^2 p$, and $\frac{\varepsilon^2}{8} n^2 p$, respectively, as $n \to \infty$, so that $H(S \cup C) - H(C) \geq \frac{\varepsilon^2}{8} n^2 p$ with high probability. With Lemma 3, this completes the proof of (18).

Suppose $\frac{|C|}{n} \geq \frac{1-h}{2} + \varepsilon$. Then, using the Chernoff bound and the union bound, it is not hard to check that for every node $i \in V \setminus C$, $(1-h)|N(i)| - 2|N(i) \cap C| \leq -\frac{\varepsilon}{2} np$ with high probability. With Lemma 4, this completes the proofs of (19) and Theorem 2(a).

2) Proof of Theorem 2(b)*:* Due to space limitation, we provide a sketch of the proof with $q = 0$. The rigorous proof with $q > 0$ is provided in the extended version of this paper [29]. Suppose $q = 0$, i.e., the PP graph G is m-disjoint ER graphs, $\{G_1 = (V_1, E_1), ..., G_m = (V_m, E_m)\}$. Let $C_l := V_l \cap C$. Then, from Lemma 4.3 in [5], it follows that

$$\Gamma(G; C) = \max_{l=1,...,m} \Gamma(G_l; C_l). \quad (20)$$

If $\frac{|C|}{n} \leq \frac{1-h}{2} - \varepsilon$, there must exist l' such that $\frac{C}{n} \leq \frac{1-h}{2} - \varepsilon\omega_m$, i.e., from Theorem 2(a), $\Gamma(G_{l'}; C_{l'}) = \Theta(n^2 p)$. Using (20), this completes the proof of part (i) in Theorem 2(b). In addition, if $\frac{|C|}{n} \leq \frac{1-h}{2} - \varepsilon$ and seed set C is selected by $\mathsf{Cluster}(k)$, i.e., C is the solution of (7), then for all l, $\frac{C}{n} \geq \frac{1-h}{2} + \varepsilon\omega_m$, i.e., from Theorem 2(a), $\Gamma(G_l; C_l) = \Theta((1 - h - \alpha)np)$. Using (20), this completes the proof of part (ii) in Theorem 2(b).



(a) PPfacebook graph (b) PLfacebook graph

Fig. 2: (a) PPfacebook [31]: 4039 nodes and 88234 edges and (b) PLfacebook [32]: 1899 nodes and 13838 edges

3) Proof of Theorem 2(c)*:* Let S be the subset of nodes with degree two. Then for large n, the fraction of S is $\frac{|S|}{n} = \frac{1}{2^\gamma(\zeta(\gamma)-1)}$. Hence the seed set C selected by $\mathsf{Degree}(k)$ with $k \geq (1-\xi_d)n$, includes all nodes with degree three or more, i.e., $V \setminus C \subset S$. Since G is connected, there exists a linear ordering $\underline{s}$ of S such that for all $t = 1, ..., |S|$, $|N(s_t) \cap S_t| \geq 1$, where $S_t = C \cup \{s_1, ..., s_{t-1}\}$. This implies that the linear ordering $\underline{s}$ satisfies (16). Thus Lemma 4 with (16) shows $\Gamma(C) = 0$. We note that the maximum degree of G is $O(n^{1/\gamma})$, since the number of nodes with degree $\omega(n^{1/\gamma})$ is $nf(n^{1/\gamma}) = o(n(n^{1/\gamma})^{-\gamma}) = o(1)$. Thus the proof of Theorem 2(c) is completed by Theorem 4 with $\Gamma(C) = 0$ and $d_{\max} = O(n^{1/\gamma})$.

V. Numerical Results

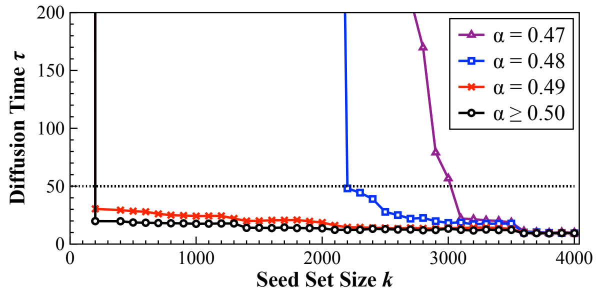
In this section, we provide simulation results based on some real social networks that demonstrate our theoretical findings.

Setup. We use two data set of the social network among Facebook users, represented as two undirected graphs, where each node corresponds to a Facebook account and each edge represents *Friend Lists* of Facebook. We name the graph from [31] PPfacebook, and the graph from [32] PLfacebook, whose graphical presentations are given in Figures 2(a) and 2(b), respectively. As hinted from the names of two graphs and Figure 2, a clustering structure is more observed in PPfacebook, whereas a skewed degree distribution (which turns out to be a power law) is prominent in PLfacebook.[4]

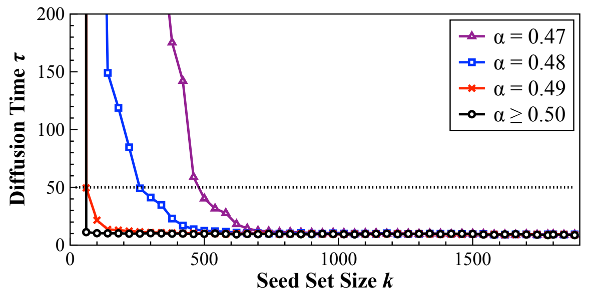
We choose $\beta = 10$ for the degree of rationality and use $h = 0.5$ for the payoff difference between the new and old products. We estimate the hitting time to $\mathbf{+1}$ for $\mathsf{General}(k)$ with varying seed set size k and diverse incentive α in each of PPfacebook and PLfacebook, where our results are averaged over 100 random samples.

Results. We plot the diffusion times in PPfacebook and PLfacebook in Figures 3(a) and 3(b), respectively, where for brevity, we omit the results with $\alpha > 0.5$ since the curve doesn't change much for $\alpha > 0.5$. As we analyzed in Section III, if the seed set size is less than a certain

[4]Our calculation reveals that the clustering coefficients of PPfacebook and PLfacebook are 0.617 and 0.1385, respectively, and the degree distributions of those two graphs are fit into power law distributions with exponent $\gamma = 1.18$ and 1.344, respectively.



(a) Diffusion time in PPfacebook



(b) Diffusion time in PLfacebook

Fig. 3: Diffusion time in PPfacebook and PLfacebook for $\alpha = 0.5, 0.49, 0.48, 0.47$ with varying seed size

value, which is 190 and 30 in PPfacebook and PLfacebook, respectively, the diffusion time is not reduced by increasing incentive α more than 0.5, while if the seed set size is sufficiently large, the diffusion time is dramatically decreased by a slight increment of incentive α from 0.47 to 0.5. Comparing Figures 3(a) and 3(b), the diffusion time reduction by incentive α in PPfacebook is more significant than the reduction in PLfacebook. Such topology-dependent impact of incentive α is analogous to our analysis of the diffusion exponents in PP and PL graphs, each of which is $\Theta((1-h-\alpha)np)$ and $O((1-h-\alpha)n^{1/\gamma})$, respectively, where $n \gg n^{1/\gamma}$. We comment that the above tendencies are similarly observed with different choice of parameters β and h, which are omitted due to space limitation.

VI. Conclusion

We model an important feature of seeded users who adjust the degree of their willingness to diffusion depending on how much incentive is given. In our model, our main question is how many seeds, i.e., quantity, and how much incentive, i.e., quality, are necessary and sufficient for accelerating the diffusion significantly. We found the phase transition of the diffusion time between the necessary and sufficient seed set sizes. Our results imply that after seeding a certain number of individuals, it is better to give more incentives to already selected seeded people instead of making efforts on seeding new ones.

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